THE COLLECTED

MATHEMATICAL WORKS

OF

GEORGE WILLIAM HILL

VOLUME TWO

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PUBLISHED BY THE CARNEGIE INSTITUTION OF WASHINGTON
FEBRUARY, 1906

CARNEGIE INSTITUTION OF WASHINGTON

PUBLICATION No. 9 (VOLUME TWO)

The Lord Baltimore Press
THE FRIEDENWALD COMPANY
BALTIMORE, MD., U. S. A.

CONTENTS

ERRATA IN SECOND VOLUME.

(Lines counted from the bottom of the page are noted as negative.)

Page	Line	
10	15	for $G' \alpha'' \gamma''$ read $G'' \alpha'' \gamma''$
13	16	for G in denom. read G'
14	4	for $G-G'$ in denom. read $G-G''$
16	9	for $\cos T$ read $\cos^2 T$
19	4	for NG'^2 read $N'G'^2$
24	15	for $\log k$ read $\log k'$
43	6	for $\nu -$ read $\nu =$
56	17	for $[n-2]$ read $[n-2]_i$
59	9	for s_{+1} read $s_{\nu+1}$
70	−5	for -0τ read -2τ
71	16	for 3^2 read $3m^2$
71	16	for $\mathrm{m}\frac{3}{2}^2$ read $\frac{3}{2}\mathrm{m}^2$
75	9	for $b_5^{(0)}$ read $b_{\frac{5}{2}}^{(0)}$
108	7	for $m\rho^2 u$ read $m'\rho^2 u$
108	19	for $h_1 \frac{d}{dt}$ read $h_1 \frac{ds}{dt}$
110	13	for $a_2 - a$ read $a_2 - a_1$
122	11	for $2\psi + 2\mathrm{a}_i$ read $2\psi + a_i$
150	8	for $s - \omega'$ read $s' - \omega'$
152	16	for $E_{-}^{(j)}$ read $E_{-i}^{(j)}$
164	11	for $\sin\varphi$ read $\sin\varphi'$
175	−2	for $e' \frac{\sin}{\cos} \tilde{\omega}$ read $e' \frac{\sin}{\cos} \tilde{\omega}'$

[INSERT IN SECOND VOLUME MATHEMATICAL WORKS OF G. W. HILL.]

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OF

GEORGE WILLIAM HILL

Ἄστρων κάτοιδα νυκτέρων ὁμήγυριν.—Æschylus.

THE COLLECTED MATHEMATICAL WORKS OF G. W. HILL

VOLUME II

MEMOIR No. 37.

On Gauss's Method of Computing Secular Perturbations, with an Application to the Action of Venus on Mercury.

(Astronomical Papers of the American Ephemeris, Vol. I, pp. 315-361. 1882.)

In 1818 Gauss presented to the Royal Society of Sciences at Göttingen a memoir, the full title of which is *Determinatio Attractionis quam in punctum quodvis positionis datae exerceret planeta si ejus massa per totam orbitam ratione temporis quo singulae partes describuntur uniformiter esset dispertita*. (*Werke, Band* III, *s*. 331.)

This memoir is a notable one in the history of elliptic functions, as it contains a new algorithm for the computation of the complete functions of Legendre's first and second species. But we shall at present view it from the side of celestial mechanics. Gauss investigates the expressions for the components of the attraction of a certain species of elliptic ring on a point, which can be advantageously employed in computing the secular perturbations of a planet, at least the parts of them which are of the first order with respect to the disturbing forces. This method merits attention because, with it we can secure almost absolute accuracy at the cost of a comparatively small outlay of labor. Moreover, it is capable of being applied, with success, to all the asteroids, and even to such refractory cases as the periodic comets. Yet, I can find but two published investigations where it has been employed. The first, a computation of the secular perturbations of the earth by Nicolai, results only being given (*Berliner Astronomische Jahrbuch für* 1820). The second, an application of the method to Tuttle's periodic comet by Dr. Thomas Clausen (*Dorpater Beobachtungen, Band* XVI, *Einleitung*). This,

perhaps, is due to the circumstance that the memoir of Gauss does not contain all the formulæ needed in the application. A double integration being necessary, Gauss has considered only that in respect to the eccentric anomaly of the disturbing body, and, having regard to elegance only, has not reduced his equations to the forms giving the utmost brevity of calculation. Hence, I propose to give an exposition of the method with the additional formulæ required.

The following notation will be adopted: For the quantities pertaining to the disturbed planet, let

a denote the semi-axis major,
n " " mean motion in a Julian year,
e " " eccentricity,
ϕ " " angle of the eccentricity, such that $e = \sin \phi$,
π " " longitude of the perihelion measured from a fixed equinox,
i " " inclination of the orbit to a fixed ecliptic,
Ω " " longitude of the ascending node of the orbit on the fixed ecliptic,
L " " mean longitude at the epoch,
χ " " longitude of the perihelion measured from a point fixed in the shifting plane of the orbit,
ω " " angular distance of the perihelion from the ascending node $= \pi - \Omega$,
r " " radius vector,
M, E, v " " mean, eccentric, and true anomalies,
u " " argument of the latitude $= v + \omega$,
m " " mass of the planet, the sun's being taken as the unit,
p " " semi-parameter $= a(1 - e^2)$.

The similar quantities belonging to the disturbing planet will be denoted by the same letters accented. In addition, let R denote the component of the disturbing force in the direction of the radius vector, positive outward from the sun; S the component of the same perpendicular to the radius vector and in the plane of the orbit, positive in the direction of motion; and W the component perpendicular to the plane of the orbit, positive northward.

The differential equations, which give the variations of the elements of the disturbed planet, are

$$\frac{da}{dt} = \frac{2a^2n \sec\varphi}{1+m}\left[e \sin v \,.\, R + \frac{p}{r} S\right]$$
$$\frac{de}{dt} = \frac{a^2n \cos\varphi}{1+m}\left[\sin v \,.\, R + (\cos v + \cos E)\, S\right]$$
$$e\frac{d\chi}{dt} = \frac{a^2n \cos\varphi}{1+m}\left[-\cos v \,.\, R + \left(\frac{r}{p}+1\right)\sin v \,.\, S\right]$$
$$\frac{di}{dt} = \frac{an \sec\varphi}{1+m}\, r \cos u \,.\, W$$
$$\sin i\frac{d\Omega}{dt} = \frac{an \sec\varphi}{1+m}\, r \sin u \,.\, W$$
$$\frac{d\pi}{dt} = \frac{d\chi}{dt} + 2\sin^2\frac{i}{2}\,.\,\frac{d\Omega}{dt}$$
$$\frac{dL}{dt} = -\frac{2an}{1+m}\, r R + 2\sin^2\frac{\varphi}{2}\,.\,\frac{d\chi}{dt} + 2\sin^2\frac{i}{2}\,.\,\frac{d\Omega}{dt} - \frac{3}{2}\int\frac{n}{a}\frac{da}{dt}dt$$

where R, S, and W involve the factor m' = the mass of the disturbing planet measured with the sun's mass as the unit, but are not multiplied by the factor k^2 (k being usually known as the Gaussian constant).*

Provided the orbits do not intersect, and if we limit the approximation to terms of the first order with respect to the disturbing forces, each of these differential coefficients can be expanded in a periodic series of the form

$$\Sigma \,.\, A \frac{\sin}{\cos}(jM + j'M')$$

j and j' being positive or negative integers, and A being constant. The term, for which both $j = 0$ and $j' = 0$, constitutes the secular portion of the series. The part of any differential coefficient, as $\frac{de}{dt}$, independent of M', is given by the definite integral

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{de}{dt}dM'$$

and the secular portion, which is independent of both M and M', by the definite integral

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{de}{dt}dMdM'.$$

But as we have the equations

$$dM = \frac{r}{a}dE = \frac{r^2}{a^2\cos^2\varphi}dv$$
$$dM' = \frac{r'}{a'}dE' = \frac{r'^2}{a'^2\cos^2\varphi'}dv'$$

* For the proof of these formulæ the reader may consult either of the following sources: Encke, *Berliner Astronomische Jahrbuch für* 1837 *und* 1838, in the treatise *Über die Berechnung der Speciellen Störungen*, which has been reprinted in Encke's *Abhandlungen*; or Oppolzer, *Lehrbuch zur Bahnbestimmung der Cometen und Planeten*, Band II, s. 218 *et seq.*; or Watson, Theoretical Astronomy, pp. 516–523.

and as the variables M, E, and v all take the values 0 and 2π together, it is possible to make the integrations with reference to the eccentric or the true anomalies of the planets. Thus we have choice between four different procedures. That in which both of the integrations are executed with reference to the eccentric anomalies is to be preferred; for the inequalities of distribution of a series of points on an elliptic orbit, corresponding to equal intervals in the value of the eccentric anomaly, are of the order of the square of the eccentricity; while, for the other two anomalies, they are of the order of the first power of this quantity. Hence, to get the secular portion of the variation of any element, as e, we shall employ the double integral

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{de}{dt}\frac{r}{a}\frac{r'}{a'}\,dE\,dE'$$

the value of which we shall denote by $\left[\frac{de}{dt}\right]_{00}$.

As, in this method, the integration, with reference to E, will be performed by quadratures, instead of the notation

$$\frac{1}{2\pi}\int_0^{2\pi} X\,dE$$

we shall use $M_E[X]$, which will denote the average of all the values of X with respect to the variable E. In the application of this method to the eight large planets of the solar system, the taking the average of 12 values, evenly distributed about the circumference with reference to E, will give, in all cases, extremely accurate results; and often 8 values will suffice. It can readily be shown, but, for the sake of brevity, we omit the demonstration, that, if the number of these values be even, the order of the error committed in the determination of the secular portions of the differential coefficients $\frac{de}{dt}$, $e\frac{d\pi}{dt}$, $\frac{di}{dt}$, and $\sin i\frac{d\Omega}{dt}$ will be the same as that of a power of the eccentricities or mutual inclination of orbits, whose exponent is one less than the number of these values, while the error, in the case of $\frac{dL}{dt}$, is of an order one degree higher. From this principle it can be judged, in any particular case, how many values ought to be computed.

It is well known that, not only when the approximation is limited to terms of the first order with respect to the disturbing forces, but even when terms of the second order are included, the secular portion of $\frac{da}{dt}$ vanishes. Hence, we can dispense with computing it.

If we put

$$
\begin{aligned}
R_0 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{ar}{m'} R\,(1 - e' \cos E')\, dE' \\
S_0 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{ar}{m'} S(1 - e' \cos E')\, dE' \\
W_0 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2}{m'} W(1 - e' \cos E')\, dE'
\end{aligned}
$$

we shall have, for the secular portions of the differential coefficients of the elements of m, the equations

$$
\begin{aligned}
\left[\frac{da}{dt}\right]_{00} &= 0 \\
\left[\frac{de}{dt}\right]_{00} &= \frac{m'n}{1+m} \cos\varphi\,.\,M_E\Big[\sin v\,.\,R_0 + (\cos v + \cos E)\,S_0\Big] \\
e\left[\frac{d\chi}{dt}\right]_{00} &= \frac{m'n}{1+m} \cos\varphi\,.\,M_E\Big[-\cos v\,.\,R_0 + \left(\frac{r}{a\cos^2\varphi} + 1\right)\sin v\,.\,S_0\Big] \\
\left[\frac{di}{dt}\right]_{00} &= \frac{m'n}{1+m} \sec\varphi\,.\,M_E\Big[\cos u\,.\,W_0\Big] \\
\sin i\left[\frac{d\Omega}{dt}\right]_{00} &= \frac{m'n}{1+m} \sec\varphi\,.\,M_E\Big[\sin u\,.\,W_0\Big] \\
\left[\frac{d\pi}{dt}\right]_{00} &= \left[\frac{d\chi}{dt}\right]_{00} + 2\sin^2\frac{i}{2}\,.\left[\frac{d\Omega}{dt}\right]_{00} \\
\left[\frac{dL}{dt}\right]_{00} &= \frac{m'n}{1+m} M_E\Big[-2\frac{r}{a}R_0\Big] + 2\sin^2\frac{\varphi}{2}\,.\left[\frac{d\chi}{dt}\right]_{00} + 2\sin^2\frac{i}{2}\,.\left[\frac{d\Omega}{dt}\right]_{00}.
\end{aligned}
$$

In the case of the earth, as the ecliptic is usually assumed as the plane of reference, at the epoch i vanishes and Ω is indeterminate. But this inconvenience is avoided by substituting for i and Ω two variables p and q (where the reader is asked not to confound this p with the p which denotes the semi-parameter), such that

$$p = \sin i \sin \Omega \qquad\qquad q = \sin i \cos \Omega\,.$$

When we shall have

$$
\begin{aligned}
\left[\frac{dp}{dt}\right]_{00} &= \frac{m'n}{1+m} \sec\varphi\,.\,M_E\Big[\sin(v+\pi)\,.\,W_0\Big] \\
\left[\frac{dq}{dt}\right]_{00} &= \frac{m'n}{1+m} \sec\varphi\,.\,M_E\Big[\cos(v+\pi)\,.\,W_0\Big].
\end{aligned}
$$

The parts of R, S, and W, which arise from the action of the disturbing planet on the sun, have, in their periodic developments, no terms independent of M'. For

$$\int \frac{x'}{r'^3}\, dM' = -\frac{n'}{1+m'}\int \frac{d^2x'}{dt^2}\, dt = -\frac{n'}{1+m'}\frac{dx'}{dt}$$

which, as it has the same value for $M' = 0$ and $M' = 2\pi$, leads to

$$\int_0^{2\pi} \frac{x'}{r'^3}\, dM' = 0\,.$$

In like manner

$$\int_0^{2\pi} \frac{y'}{r'^3}\, dM' = 0 \qquad\qquad \int_0^{2\pi} \frac{z'}{r'^3}\, dM' = 0\,.$$

Hence, for our present purpose, it will suffice to consider only the mutual action of the two planets. Then, assuming a system of rectangular co-ordinates, two of whose axes, x and y, lie in the plane of the orbit of the disturbed planet, so that $z = 0$, R, S, and W are determined by the equations

$$\frac{r}{m'} R = \frac{xx' + yy' - r^2}{\Delta^3}$$
$$\frac{r}{m'} S = \frac{xy' - x'y}{\Delta^3}$$
$$\frac{1}{m'} W = \frac{z'}{\Delta^3}$$

and the distance Δ of the two planets by the equation

$$\Delta^2 = r^2 - 2\,(xx' + yy') + r'^2.$$

In order to accomplish the integrations which R_0, S_0, and W_0 involve, it will be necessary to express R, S, and W explicitly in terms of the variable E'. If I denotes the mutual inclination of the orbits, and Π and Π' severally the angular distances of the perihelia from the ascending node of the orbit of the disturbing planet on the orbit of the disturbed, these quantities are determined by the equations

$$\begin{aligned}
\sin I \cos (\Pi - \omega) &= -\sin i \cos i' + \cos i \sin i' \cos (\Omega' - \Omega)\\
\sin I \sin (\Pi - \omega) &= -\sin i' \sin (\Omega' - \Omega)\\
\sin I \cos (\Pi' - \omega') &= \cos i \sin i' - \sin i \cos i' \cos (\Omega' - \Omega)\\
\sin I \sin (\Pi' - \omega') &= -\sin i \sin (\Omega' - \Omega).
\end{aligned}$$

We shall then have

$$\begin{aligned}
xx' + yy' &= rr' \left[\cos (v + \Pi) \cos (v' + \Pi') + \cos I \sin (v + \Pi) \sin (v' + \Pi')\right]\\
xy' - x'y &= rr' \left[-\sin (v + \Pi) \cos (v' + \Pi') + \cos I \cos (v + \Pi) \sin (v' + \Pi')\right]\\
z' &= r' \sin I \sin (v' + \Pi').
\end{aligned}$$

But if four auxiliary constants, k, K, k', and K', are so taken that

$$k \cos (K - \Pi) = \cos \Pi' \qquad\qquad k' \cos (K' - \Pi) = \cos I \cos \Pi'$$
$$k \sin (K - \Pi) = -\cos I \sin \Pi' \qquad\qquad k' \sin (K' - \Pi) = -\sin \Pi'$$

the first two equations take the forms

$$\begin{aligned} xx' + yy' &= \quad kr\cos(v+K).r'\cos v' + k'r\sin(v+K').r'\sin v' \\ xy' - x'y &= -kr\sin(v+K).r'\cos v' + k'r\cos(v+K').r'\sin v'. \end{aligned}$$

By the substitution of the values

$$r'\cos v' = a'(\cos E' - e') \qquad\qquad r'\sin v' = a'\cos\varphi'\sin E'$$

we have

$$\begin{aligned} xx' + yy' &= \quad ka'r\cos(v+K)\ (\cos E' - e') + k'a'\cos\varphi'.r\sin(v+K')\sin E' \\ xy' - x'y &= -ka'r\sin(v+K)\ (\cos E' - e') + k'a'\cos\varphi'.r\cos(v+K')\sin E' \\ z' &= \quad a'\sin I\sin\Pi'(\cos E' - e') + a'\sin I\cos\Pi'\cos\varphi'\sin E'. \end{aligned}$$

Moreover,

$$r' = a'(1 - e'\cos E')$$

in consequence, if we put

$$\begin{aligned} A &= r^2 + 2ka'e'r\cos(v+K) + a'^2 \\ B\cos\epsilon &= ka'r\cos(v+K) + a'^2e' \\ B\sin\epsilon &= k'a'\ \cos\varphi'.r\sin(v+K') \\ C &= a'^2e'^2 \end{aligned}$$

we shall have

$$\Delta^2 = A - 2B\cos(E' - \epsilon) + C\cos^2 E'.$$

R, S, and W are now expressed explicitly in terms of E'. Gauss's method of effecting the integrations, which give R_0, S_0, and W_0, consists in taking a new variable T, such that

$$\cos E' = \frac{\alpha + \alpha'\sin T + \alpha''\cos T}{\gamma + \gamma'\sin T + \gamma''\cos T}$$
$$\sin E' = \frac{\beta + \beta'\sin T + \beta''\cos T}{\gamma + \gamma'\sin T + \gamma''\cos T}$$

where α, β, γ, etc., satisfy certain conditions, and, moreover, are so taken that the coefficients of $\sin T$, $\cos T$ and $\sin T\cos T$ vanish in the expression

$$\Delta^2[\gamma + \gamma'\sin T + \gamma''\cos T]^2$$

which, in consequence, takes the form

$$G - G'\sin^2 T + G''\cos^2 T.$$

As the equation

$$[\alpha + \alpha'\sin T + \alpha''\cos T]^2 + [\beta + \beta'\sin T + \beta''\cos T]^2 - [\gamma + \gamma'\sin T + \gamma''\cos T]^2 = 0$$

ought to hold true independently of the value of T, the left member must have the form

$$k\,(\sin^2 T + \cos^2 T - 1)$$

but as it is plain that the values of α, α', etc., can be multiplied by a common factor without any change resulting in sin E' and cos E', we may assume $k = 1$. We then have the six equations of condition

$$\begin{array}{ll} \alpha^2 + \beta^2 - \gamma^2 = -1 & \alpha\alpha' + \beta\beta' - \gamma\gamma' = 0 \\ \alpha'^2 + \beta'^2 - \gamma'^2 = 1 & \alpha\alpha'' + \beta\beta'' - \gamma\gamma'' = 0 \\ \alpha''^2 + \beta''^2 - \gamma''^2 = 1 & \alpha'\alpha'' + \beta'\beta'' - \gamma'\gamma'' = 0. \end{array}$$

From the values of sin E' and cos E' in terms of T, by having regard to the equations of condition just written, we obtain

$$\begin{aligned} \alpha \cos E' + \beta \sin E' - \gamma &= \frac{-1}{\gamma + \gamma' \sin T + \gamma'' \cos T} \\ \alpha' \cos E' + \beta' \sin E' - \gamma' &= \frac{\sin T}{\gamma + \gamma' \sin T + \gamma'' \cos T} \\ \alpha'' \cos E' + \beta'' \sin E' - \gamma'' &= \frac{\cos T}{\gamma + \gamma' \sin T + \gamma'' \cos T}. \end{aligned}$$

Hence, as the equation

$$[\alpha \cos E' + \beta \sin E' - \gamma]^2 - [\alpha' \cos E' + \beta' \sin E' - \gamma']^2 - [\alpha'' \cos E' + \beta'' \sin E' - \gamma'']^2 = 0$$

ought to be satisfied independently of the value assigned to E', the left member must have the form

$$k\,[\sin^2 E' + \cos^2 E' - 1].$$

Consequently,

$$\begin{array}{ll} \alpha^2 - \alpha'^2 - \alpha''^2 = k & \alpha\beta - \alpha'\beta' - \alpha''\beta'' = 0 \\ \beta^2 - \beta'^2 - \beta''^2 = k & \alpha\gamma - \alpha'\gamma' - \alpha''\gamma'' = 0 \\ \gamma^2 - \gamma'^2 - \gamma''^2 = -k & \beta\gamma - \beta'\gamma' - \beta''\gamma'' = 0. \end{array}$$

But by comparing the three of these equations which involve squares of the quantities α, α', etc., with the similar three of the equations of condition previously obtained, we get $3k = -3$, or $k = -1$.

The six equations of condition first obtained may be so written as to form three groups of linear equations, thus:

$$\begin{array}{lll} \alpha\,.\,\alpha + \beta\,.\,\beta - \gamma\,.\,\gamma = -1 & \alpha\,.\,\alpha' + \beta\,.\,\beta' - \gamma\,.\,\gamma' = 0 & \alpha\,.\,\alpha'' + \beta\,.\,\beta'' - \gamma\,.\,\gamma'' = 0 \\ \alpha'\,.\,\alpha + \beta'\,.\,\beta - \gamma'\,.\,\gamma = 0 & \alpha'\,.\,\alpha' + \beta'\,.\,\beta' - \gamma'\,.\,\gamma' = 1 & \alpha'\,.\,\alpha'' + \beta'\,.\,\beta'' - \gamma'\,.\,\gamma'' = 0 \\ \alpha''\,.\,\alpha + \beta''\,.\,\beta - \gamma''\,.\,\gamma = 0 & \alpha''\,.\,\alpha' + \beta''\,.\,\beta' - \gamma''\,.\,\gamma' = 0 & \alpha''\,.\,\alpha'' + \beta''\,.\,\beta'' - \gamma''\,.\,\gamma'' = 1. \end{array}$$

If we put

$$D = \alpha\beta'\gamma'' - \alpha'\beta\gamma'' + \alpha'\beta''\gamma - \alpha''\beta'\gamma + \alpha''\beta\gamma' - \alpha\beta''\gamma'$$

we shall have

$$D\alpha = -\frac{dD}{d\alpha} = \beta''\gamma' - \beta'\gamma'' \qquad D\alpha' = \frac{dD}{d\alpha'} = \beta''\gamma - \beta\gamma''$$

$$D\beta = -\frac{dD}{d\beta} = \alpha'\gamma'' - \alpha''\gamma' \qquad D\beta' = \frac{dD}{d\beta'} = \alpha\gamma'' - \alpha''\gamma$$

$$D\gamma = \frac{dD}{d\gamma} = \alpha'\beta'' - \alpha''\beta' \qquad D\gamma' = -\frac{dD}{d\gamma'} = \alpha\beta'' - \alpha''\beta$$

$$D\alpha'' = \frac{dD}{d\alpha''} = \beta\gamma' - \beta'\gamma$$

$$D\beta'' = \frac{dD}{d\beta''} = \alpha'\gamma - \alpha\gamma'$$

$$D\gamma'' = -\frac{dD}{d\gamma''} = \alpha'\beta - \alpha\beta'.$$

The value of D may be found by taking any one of the twelve preceding equations of condition between α, α', etc., and substituting in it the values of α, α', etc., from the preceding nine equations. Thus, if we take the equation

$$\alpha^2 - \alpha'^2 - \alpha''^2 = -1$$

we shall have

$$\begin{aligned} D^2(-\alpha^2 + \alpha'^2 + \alpha''^2) = D^2 &= (\beta\gamma' - \beta'\gamma)^2 + (\beta''\gamma - \beta\gamma'')^2 - (\beta'\gamma'' - \beta''\gamma')^2 \\ &= \beta^2\gamma'^2 + \beta'^2\gamma^2 + \beta''^2\gamma^2 + \beta^2\gamma''^2 - \beta'^2\gamma''^2 - \beta''^2\gamma'^2 \\ &\quad - 2\beta\gamma\beta'\gamma' - 2\beta\gamma\beta''\gamma'' + 2\beta'\gamma'\beta''\gamma'' \\ &= \beta^2(\gamma^2 - 1) + \beta'^2(\gamma'^2 + 1) + \beta''^2(\gamma''^2 + 1) \\ &\quad - 2\beta\gamma\beta'\gamma' - 2\beta\gamma\beta''\gamma'' + 2\beta'\gamma'\beta''\gamma'' \\ &= -\beta^2 + \beta'^2 + \beta''^2 + (\beta\gamma - \beta'\gamma' - \beta''\gamma'')^2 \\ &= 1. \end{aligned}$$

Hence, $D = \pm 1$. It is evident we may adopt either sign, consequently we take the positive one.

The foregoing equations between the quantities α, α', etc., are all that are necessary for our purposes, but in order to obtain the values of these quantities and also of the three G, G', and G'' we must have recourse to the equations furnished by the tronsformation of the expression for Δ^2. This transformation evidently comes to the same thing as the changing of the expression

$$Az^2 - 2B\cos\epsilon \,.\, xz - 2B\sin\epsilon \,.\, yz + Cx^2$$

into

$$Gu^2 - G'u'^2 + G''u''^2$$

by the employment of the formulæ

$$
\begin{aligned}
x &= \alpha u + \alpha' u' + \alpha'' u'' \\
y &= \beta u + \beta' u' + \beta'' u'' \\
z &= \gamma u + \gamma' u' + \gamma'' u''.
\end{aligned}
$$

But, having regard to the equations which the quantities α, α', etc., satisfy, we readily deduce from the last-given equations

$$
\begin{aligned}
u &= -\alpha x - \beta y + \gamma z \\
u' &= \alpha' x + \beta' y - \gamma' z \\
u'' &= \alpha'' x + \beta'' y - \gamma'' z.
\end{aligned}
$$

By substitution of these values in the expression $Gu^2 - G'u'^2 + G''u''^2$ and comparison of the resulting coefficients with

$$Az^2 - 2B \cos \varepsilon \,.\, xz - 2B \sin \varepsilon \,.\, yz + Cx^2$$

we get the following equations:

$$
\begin{aligned}
G\alpha^2 - G'\alpha'^2 + G''\alpha''^2 &= C & \qquad G\alpha\beta - G'\alpha'\beta' + G''\alpha''\beta'' &= 0 \\
G\beta^2 - G'\beta'^2 + G''\beta''^2 &= 0 & \qquad G\alpha\gamma - G'\alpha'\gamma' + G''\alpha''\gamma'' &= B \cos \varepsilon \\
G\gamma^2 - G'\gamma'^2 + G''\gamma''^2 &= A & \qquad G\beta\gamma - G'\beta'\gamma' + G''\beta''\gamma'' &= B \sin \varepsilon
\end{aligned}
$$

which, in conjunction with the six independent equations between α, α', etc., previously obtained, suffice to determine the twelve unknowns, $\alpha, \alpha', \alpha''$, $\beta, \beta', \beta'', \gamma, \gamma', \gamma''$, G, G', and G''.

These six equations can be written in three groups of three equations each, the first group being as follows:

$$
\begin{aligned}
\alpha \,.\, G\alpha - \alpha' \,.\, G'\alpha' + \alpha'' \,.\, G''\alpha'' &= C \\
\alpha \,.\, G\beta - \alpha' \,.\, G'\beta' + \alpha'' \,.\, G''\beta'' &= 0 \\
\alpha \,.\, G\gamma - \alpha' \,.\, G'\gamma' + \alpha'' \,.\, G''\gamma'' &= B \cos \varepsilon.
\end{aligned}
$$

The second and third groups are obtained from this by writing in succession β and γ for α in the first factors of the terms of the left members of the equations, and making the second members, in the first case, severally 0, 0, and $B \sin \varepsilon$, and in the second, $B \cos \varepsilon$, $B \sin \varepsilon$, and A. By having regard to the six equations of condition between α, α', etc., which were first obtained, we get from these three groups severally the following three groups of equations:

$$
\left\{
\begin{aligned}
G\alpha &= -C\alpha + B \cos \varepsilon \,.\, \gamma \\
G\beta &= B \sin \varepsilon \,.\, \gamma \\
G\gamma &= -B \cos \varepsilon \,.\, \alpha - B \sin \varepsilon \,.\, \beta + A\gamma
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
G'\alpha' &= -C\alpha' + B \cos \varepsilon \,.\, \gamma' \\
G'\beta' &= B \sin \varepsilon \,.\, \gamma' \\
G'\gamma' &= -B \cos \varepsilon \,.\, \alpha' - B \sin \varepsilon \,.\, \beta' + A\gamma'
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
-G''\alpha'' &= -C\alpha'' + B \cos \varepsilon \,.\, \gamma'' \\
-G''\beta'' &= B \sin \varepsilon \,.\, \gamma'' \\
-G''\gamma'' &= -B \cos \varepsilon \,.\, \alpha'' - B \sin \varepsilon \,.\, \beta'' + A\gamma''.
\end{aligned}
\right.
$$

From the first two equations of each of these three groups is obtained

$$\alpha = \frac{B\cos\epsilon}{G+C}\gamma \qquad \alpha' = \frac{B\cos\epsilon}{G'+C}\gamma' \qquad \alpha'' = \frac{B\cos\epsilon}{C-G''}\gamma''$$
$$\beta = \frac{B\sin\epsilon}{G}\gamma \qquad \beta' = \frac{B\sin\epsilon}{G'}\gamma' \qquad \beta'' = -\frac{B\sin\epsilon}{G''}\gamma''.$$

By substituting these values of α, β, etc., in the last equation of each group we obtain

$$G - A + \frac{B^2\cos^2\epsilon}{G+C} + \frac{B^2\sin^2\epsilon}{G} = 0$$
$$G' - A + \frac{B^2\cos^2\epsilon}{G'+C} + \frac{B^2\sin^2\epsilon}{G'} = 0$$
$$-G'' - A + \frac{B^2\cos^2\epsilon}{-G''+C} + \frac{B^2\sin^2\epsilon}{-G''} = 0.$$

It is evident, now, that G, G', and $-G''$ are the roots of the cubic equation

$$x - A + \frac{B^2\cos^2\epsilon}{x+C} + \frac{B^2\sin^2\epsilon}{x} = 0$$

or of

$$x[(x-A)(x+C)+B^2] + B^2C\sin^2\epsilon = 0.$$

The roots of this equation are all real, as can be shown in the following manner: If, for the moment, we adopt Gauss's system of rectangular co-ordinates, that is, put the origin at the center of the ellipse described by the disturbing planet, and make the axes of x and y coincide severally with the major and minor axes of this ellipse, and suppose that the co-ordinates of the disturbed planet, with reference to this system of axes are denoted by A, B, and C, the expression for Δ^2, which, in our notation, is

$$\Delta^2 = A - 2B\cos(E'-\epsilon) + C\cos^2 E'$$

will become

$$\begin{aligned}\Delta^2 &= (A - a'\cos E')^2 + (B - a'\cos\varphi'\sin E')^2 + C^2\\ &= A^2 + B^2 + C^2 + a'^2\cos^2\varphi' - 2(Aa'\cos E' + Ba'\cos\varphi'\sin E') + a'^2\sin^2\varphi'\cos^2 E'.\end{aligned}$$

By comparison of these two expressions for Δ^2, we find that, expressed in terms of the second system of co-ordinates, the equation in x becomes

$$\begin{aligned}x[x-(A^2+B^2+C^2+a'^2\cos^2\varphi')](x+a'^2\sin^2\varphi') + (A^2a'^2+B^2a'^2\cos^2\varphi')x\\ + B^2a'^4\sin^2\varphi'\cos^2\varphi' = 0.\end{aligned}$$

We substitute for x in this equation the four values $-C$, 0, $a'^2\cos^2\phi'$, and A, and obtain the results

$$
\begin{array}{llll}
x = & -a'^2\sin^2\varphi' = -C & \text{result,} & -A^2a'^4\sin^2\varphi' \\
x = & 0 & \text{"} & +B^2a'^4\sin^2\varphi'\cos^2\varphi' \\
x = & a'^2\cos^2\varphi' & \text{"} & -C^2a'^4\cos^2\varphi' \\
x = & A & \text{"} & +B^2(A+C\sin^2\varepsilon).
\end{array}
$$

From this it is apparent that the roots are all real, one being negative and numerically less than C, one positive and less than $a'^2\cos^2\phi'$, and another positive and lying between $a'^2\cos^2\phi'$ and A.

The assignment of these roots as the values of G, G', and $-G''$ is not indifferent; as we wish both Δ and the transformation to be real, we put G equal to the larger of the positive roots, G' equal to the smaller, and $-G''$ equal to the negative root. Consequently, G, G', and G'' are always positive quantities.

The readiest method of obtaining them from the equation of the third degree, which determines them, appears to by trial. If we put

$$
\begin{aligned}
g &= B^2C\sin^2\varepsilon \\
h &= \tfrac{1}{2}[A - C + \sqrt{(A+C)^2 - 4B^2}] \\
l &= \tfrac{1}{2}[A - C - \sqrt{(A+C)^2 - 4B^2}]
\end{aligned}
$$

the equation takes the form

$$x(x-h)(x-l) + g = 0.$$

As g is usually a small quantity, having the factor e'^2, the approximate values of the roots are 0, l, and h. G, G', and G'' can then be obtained, by successive approximations, from the equation put in the forms

$$
\begin{aligned}
G &= h - \frac{g}{G(G-l)} \\
G' &= l + \frac{g}{G'(h-G')} \\
G'' &= \frac{g}{(h+G'')(l+G'')}
\end{aligned}
$$

quite approximate values being

$$G = h - \frac{g}{h(h-l)} \qquad G' = l + \frac{g}{l(h-l)} \qquad G'' = \frac{g}{\left(h+\frac{g}{hl}\right)\left(l+\frac{g}{hl}\right)}.$$

For verification we may employ either or both of the equations

$$G + G' - G'' = A - C$$
$$GG'G'' = B^2 C \sin^2 \varepsilon.$$

It will be seen that, in order to make our desired transformation from the variable E to the variable T, we do not need the values of the nine quantities α, α', etc., but only the values of the following ten squares and products of them, viz., α'^2, γ'^2, $\alpha'\beta'$, $\alpha'\gamma'$, $\beta'\gamma'$, α''^2, γ''^2, $\alpha''\beta''$, $\alpha''\gamma''$, and $\beta''\gamma''$; hence, we will limit ourselves to the determination of these.

The values of α' and β', in terms of γ', and of α'' and β'', in terms of γ'', have already been given. If we substitute them in the equations

$$\alpha'^2 + \beta'^2 - \gamma'^2 = 1 \qquad\qquad \alpha''^2 + \beta''^2 - \gamma''^2 = 1$$

we obtain

$$\left[\frac{B^2 \cos^2 \varepsilon}{(G' + C)^2} + \frac{B^2 \sin^2 \varepsilon}{G'^2} - 1\right] \gamma'^2 = 1$$
$$\left[\frac{B^2 \cos^2 \varepsilon}{(C - G'')^2} + \frac{B^2 \sin^2 \varepsilon}{G''^2} - 1\right] \gamma''^2 = 1.$$

Whence

$$\gamma'^2 = \frac{(G' + C)\, G'}{\dfrac{B^2 \cos^2 \varepsilon}{G' + C}\, G + \dfrac{B^2 \sin^2 \varepsilon}{G'}(G' + C) - (G' + C)\, G'}$$

or having regard to the equation which determines G',

$$\gamma'^2 = \frac{(G' + C)\, G'}{(A - G')\, G' + \dfrac{B^2 C \sin^2 \varepsilon}{G'} - (G' + C)\, G'}$$
$$= \frac{(G' + C)\, G'}{(A - C - 2G')\, G' + GG''}$$
$$= \frac{(G' + C)\, G'}{(G' + G'')(G - G')}.$$

And in like manner,

$$\gamma''^2 = \frac{(C - G'')\, G''}{\dfrac{B^2 \cos^2 \varepsilon}{C - G''}\, G'' + \dfrac{B^2 \sin^2 \varepsilon}{G''}(C - G'') - (C - G'')\, G''}$$
$$= \frac{(C - G'')\, G''}{(A + G'')\, G'' + GG' - (C - G'')\, G''}$$
$$= \frac{(C - G'')\, G''}{(G + G'')(G' + G'')}.$$

We have

$$\frac{B^2 \cos^2 \epsilon}{G' + C} = A - G' - \frac{B^2 \sin^2 \epsilon}{G'}$$

consequently,

$$\alpha'^2 = \frac{(A - G')\, G' - B^2 \sin^2 \epsilon}{(G' + G'')(G - G')}\,.$$

Also,

$$\frac{B^2 \cos^2 \epsilon}{C - G''} = A + G'' + \frac{B^2 \sin^2 \epsilon}{G''}$$

consequently,

$$\alpha''^2 = \frac{(A + G'')\, G'' + B^2 \sin^2 \epsilon}{(G + G'')(G' + G'')}\,.$$

And the values of the six products needed are

$$\alpha'\beta' = \frac{B^2 \sin \epsilon \cos \epsilon}{(G' + G'')(G - G')} \qquad \alpha''\beta'' = -\frac{B^2 \sin \epsilon \cos \epsilon}{(G + G'')(G' + G'')}$$

$$\alpha'\gamma' = \frac{B \cos \epsilon .\, G'}{(G' + G'')(G - G')} \qquad \alpha''\gamma'' = \frac{B \cos \epsilon .\, G''}{(G + G'')(G' + G'')}$$

$$\beta'\gamma' = \frac{B \sin \epsilon .\, (C + G')}{(G' + G'')(G - G')} \qquad \beta''\gamma'' = -\frac{B \sin \epsilon .\, (C - G'')}{(G + G'')(G' + G'')}\,.$$

We have next to ascertain the value of the differential dE' in terms of the differential dT. From the equations

$$H \cos E' = \alpha + \alpha' \sin T + \alpha'' \cos T$$
$$H \sin E' = \beta + \beta' \sin T + \beta'' \cos T$$

where H stands for $\gamma + \gamma' \sin T + \gamma'' \cos T$, it follows that

$$HdE' = [\cos E' (\beta' \cos T - \beta'' \sin T) - \sin E' (\alpha' \cos T - \alpha'' \sin T)]\, dT$$

or

$$H^2 dE' = [(\alpha''\beta' - \alpha'\beta'') + (\alpha''\beta - \alpha\beta'') \sin T + (\alpha\beta' - \alpha'\beta) \cos T]\, dT$$
$$= -[\gamma + \gamma' \sin T + \gamma'' \cos T]\, dT.$$

Whence

$$HdE' = -dT.$$

The quantity H is always of the same sign, otherwise sin E' and cos E' might become infinite in the passage of H through zero. If this consideration is not deemed conclusive, the point can be established as follows:

Since we have

$$(\gamma' \sin T + \gamma'' \cos T)^2 + (\gamma'' \sin T - \gamma' \cos T)^2 = \gamma'^2 + \gamma''^2 = \gamma^2 - 1$$

without regard to signs, $\gamma' \sin T + \gamma'' \cos T$ will always be less than γ. Hence, if γ be negative, T will always increase when E' increases; but if γ be positive, T will always diminish when E increases.

If we put $\sqrt{\gamma^2 - 1} = \delta$, so that $\delta^2 = \alpha^2 + \beta^2 = \gamma'^2 + \gamma''^2$, we shall have:

$$\begin{aligned} H(\delta + \alpha \cos E' + \beta \sin E') &= \gamma\delta + \alpha^2 + \beta^2 + (\gamma'\delta + \alpha\alpha' + \beta\beta') \sin T \\ &\quad + (\gamma''\delta + \alpha\alpha'' + \beta\beta'') \cos T \\ &= (\gamma + \delta)(\delta + \gamma' \sin T + \gamma'' \cos T). \end{aligned}$$

Also,

$$\begin{aligned} H(\alpha \sin E' - \beta \cos E') &= (\alpha\beta' - \alpha'\beta) \sin T + (\alpha\beta'' - \alpha''\beta) \cos T \\ &= -\gamma'' \sin T + \gamma' \cos T. \end{aligned}$$

By putting

$$\frac{\alpha}{\delta} = \cos L \qquad \frac{\beta}{\delta} = \sin L \qquad \frac{\gamma''}{\delta} = \cos M \qquad \frac{\gamma'}{\delta} = \sin M$$

these two equations become

$$\begin{aligned} H[1 + \cos(E' - L)] &= (\gamma + \delta)[1 + \cos(T - M)] \\ H \sin(E' - L) &= -\sin(T - M). \end{aligned}$$

By division we get

$$\tan \tfrac{1}{2}(T - M) = -(\gamma + \delta) \tan \tfrac{1}{2}(E' - L).$$

From this equation it is evident that, when E' augments by a circumference, T augments or diminishes by the same quantity according as γ is negative or positive.

The expressions we have to integrate with respect to E' are of the form $\frac{\Theta}{\Delta^3}$; hence, whether γ be positive or negative, we shall always have

$$\int_0^{2\pi} \frac{\theta}{\Delta^3} dE' = \int_0^{2\pi} \frac{H^2\theta}{(H^2\Delta^2)^{\frac{3}{2}}} dT$$

provided that we understand that the radical in the denominator is to have the positive sign.

The general form of Θ is

$$\begin{aligned} \theta &= [f + g(\cos E' - e') + h \sin E'](1 - e' \cos E') \\ &= f - ge' + [g(1 + e'^2) - fe'] \cos E' + h \sin E' - he' \sin E' \cos E' - ge' \cos^2 E'. \end{aligned}$$

If in this expression, multiplied by H^2, are substituted the values of H^2, $H \cos E$, and $H \sin E$ in terms of T, and the terms multiplied by $\sin T$, $\cos T$, and $\sin T \cos T$ omitted, as, when integrated between the limits 0 and 2π they contribute nothing to the value of the integral, we get

$$\begin{aligned}
H^2\theta = &(f - ge')(\gamma^2 + \gamma'^2 \sin^2 T + \gamma''^2 \cos^2 T) \\
&+ [g(1 + e'^2) - fe'](\alpha\gamma + \alpha'\gamma' \sin^2 T + \alpha''\gamma'' \cos^2 T) \\
&+ h(\beta\gamma + \beta'\gamma' \sin^2 T + \beta''\gamma'' \cos^2 T) \\
&- he'(\alpha\beta + \alpha'\beta' \sin^2 T + \alpha''\beta'' \cos^2 T) \\
&- ge'(\alpha^2 + \alpha'^2 \sin^2 T + \alpha''^2 \cos T).
\end{aligned}$$

But we have the equations

$$\begin{aligned}
\alpha^2 &= -1 + \alpha'^2 + \alpha''^2 \\
\gamma^2 &= 1 + \gamma'^2 + \gamma''^2 \\
\alpha\beta &= \alpha'\beta' + \alpha''\beta'' \\
\alpha\gamma &= \alpha'\gamma' + \alpha''\gamma'' \\
\beta\gamma &= \beta'\gamma' + \beta''\gamma''.
\end{aligned}$$

Hence, if we put

$$\begin{aligned}
\Gamma' &= (f - ge')\gamma'^2 + [g(1 + e'^2) - fe']\alpha'\gamma' + h\beta'\gamma' - he'\alpha'\beta' - ge'\alpha'^2 \\
\Gamma'' &= (f - ge')\gamma''^2 + [g(1 + e'^2) - fe']\alpha''\gamma'' + h\beta''\gamma'' - he'\alpha''\beta'' - ge'\alpha''^2
\end{aligned}$$

we shall have

$$H^2\theta = [2\Gamma' + \Gamma'' + f]\sin^2 T + [\Gamma' + 2\Gamma'' + f]\cos^2 T.$$

If we substitute, in the expressions for Γ' and Γ'', for γ'^2, $\alpha'\gamma'$, etc., the values we have previously obtained for these squares and products, and, moreover, put

$$\begin{aligned}
F &= [ge' B \sin \epsilon - he' B \cos \epsilon + hC] B \sin \epsilon \\
J &= -ge'A + (f - ge') C + [g(1 + e'^2) - fe') B \cos \epsilon + hB \sin \epsilon
\end{aligned}$$

we shall obtain

$$\Gamma' = \frac{F + JG' + fG'^2}{(G' + G'')(G - G')} \qquad \Gamma'' = \frac{-F + JG'' - fG''^2}{(G + G'')(G' + G'')}.$$

Substituting in the values of F and J the values of A, $B \cos \epsilon$, $B \sin \epsilon$, and C, we get

$$\begin{aligned}
F &= a'e'r B \sin \epsilon\, [gk' \cos \varphi' \sin (v + K') - hk \cos (v + K)] \\
J &= -fa'e'kr \cos (v + K) + g[ka' \cos^2 \varphi' . r \cos (v + K) - e'r^2] \\
&\qquad + hk'a' \cos \varphi' . r \sin (v + K').
\end{aligned}$$

To apply these formulæ to the three special cases of the computation of R_0, S_0, and W_0. In the case of R_0 we have

$$f = -ar^2 \qquad g = kaa'r \cos(v + K) \qquad h = k'aa' \cos\varphi' . r \sin(v + K').$$

Consequently, here

$$\begin{aligned} F &= 0 \\ J &= aa'^2 \cos^2\varphi' . r^2 [k^2 \cos^2(v + K) + k'^2 \sin^2(v + K')] \\ &= aa'^2 \cos^2\varphi' . r^2 [1 - \sin^2 I \sin^2(v + \Pi)]. \end{aligned}$$

In the case of S_0 we have

$$f = 0 \qquad g = -kaa'r \sin(v + K) \qquad h = k'aa' \cos\varphi' . r \cos(v + K').$$

Consequently, here

$$\begin{aligned} F &= - aa'^2 kk' \cos(K' - K) \sin\varphi' \cos\varphi' . r^3 B \sin\epsilon \\ &= - aa'^2 \sin\varphi' \cos\varphi' \cos I . r^3 B \sin\epsilon \\ J &= kaa'e'r^3 \sin(v + K) + \tfrac{1}{2} aa'^2 \cos^2\varphi' . r^2 [k'^2 \sin 2(v + K') - k^2 \sin 2(v + K)] \\ &= kaa'e'r^3 \sin(v + K) - \tfrac{1}{2} aa'^2 \cos^2\varphi' \sin^2 I . r^2 \sin 2(v + \Pi). \end{aligned}$$

In the case of W_0 we have

$$f = 0 \qquad g = a' \sin I \sin \Pi' . r^2 \qquad h = a' \sin I \cos \Pi' \cos\varphi' . r^2.$$

Consequently, here

$$\begin{aligned} F &= a'^2 \sin\varphi' \cos\varphi' \sin I . r^3 B \sin\epsilon [k' \sin\Pi' \sin(v + K') - k \cos\Pi' \cos(v + K)] \\ &= - a'^2 \sin\varphi' \cos\varphi' \sin I . r^3 \cos(v + \Pi) . B \sin\epsilon \\ J &= a'^2 \cos^2\varphi' \sin I . r^3 [k \sin\Pi' \cos(v + K) + k' \cos\Pi' \sin(v + K')] \\ &\quad - a' \sin\varphi' \sin I \sin\Pi' . r^4 \\ &= a'^2 \cos^2\varphi' \sin I \cos I . r^3 \sin(v + \Pi) - a'e' \sin I \sin\Pi' . r^4. \end{aligned}$$

The values of R_0, S_0, and W_0 are given by the definite integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{[2\Gamma' + \Gamma'' + f] \sin^2 T + [\Gamma' + 2\Gamma'' + f] \cos^2 T}{[G + G'']^{\frac{3}{2}} [1 - c^2 \sin^2 T]^{\frac{3}{2}}} dT$$

provided we attribute to F, J, and f the values they have in each case. In this expression we have put

$$\frac{G' + G''}{G + G''} = c^2$$

c is then the modulus of the elliptic integrals involved in the expression. Let b denote the complementary modulus $= \sqrt{1 - c^2}$. In the notation of Legendre

$$\int_0^{\frac{\pi}{2}} \frac{dT}{[1 - c^2 \sin^2 T]^{\frac{1}{2}}} = F^1(c) \qquad \int_0^{\frac{\pi}{2}} [1 - c^2 \sin^2 T]^{\frac{1}{2}} \, dT = E^1(c).$$

We have the equation

$$\frac{d}{dT} \frac{\sin T \cos T}{[1 - c^2 \sin^2 T]^{\frac{1}{2}}} = \frac{1 - 2 \sin^2 T + c^2 \sin^4 T}{[1 - c^2 \sin^2 T]^{\frac{3}{2}}}$$

whence

$$\int_0^{\frac{\pi}{2}} \frac{1 - 2 \sin^2 T + c^2 \sin^4 T}{[1 - c^2 \sin^2 T]^{\frac{3}{2}}} dT = 0.$$

In consequence, we have the equations

$$\int_0^{\frac{\pi}{2}} \frac{(1 - c^2)\, dT}{[1 - c^2 \sin^2 T]^{\frac{3}{2}}} = E^1(c)$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 T dT}{[1 - c^2 \sin^2 T]^{\frac{3}{2}}} = \frac{1}{c^2} \left[\frac{1}{b^2} E^1(c) - F^1(c) \right]$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 T dT}{[1 - c^2 \sin^2 T]^{\frac{3}{2}}} = \frac{1}{c^2} \left[F^1(c) - E^1(c) \right].$$

Legendre, moreover, has put

$$F^1(c) = \frac{\pi}{2} K \qquad E^1(c) = \frac{\pi}{2} KL$$

Hence,

$$R_0, S_0 \text{ or } W_0 = \frac{K}{c^2 (G + G'')^{\frac{3}{2}}} \left[(\Gamma' + 2\Gamma'' + f)(1 - L) + (2\Gamma' + \Gamma'' + f)\left(\frac{L}{b^2} - 1\right) \right]$$

$$= \frac{KL}{b^2 (G + G'')^{\frac{3}{2}}} f + \frac{K}{(G + G'')^{\frac{3}{2}}} \left[\frac{L}{b^2} + \frac{L - b^2}{b^2 c^2} \right] \Gamma' + \frac{K}{(G + G'')^{\frac{3}{2}}} \left[2 \frac{L}{b^2} - \frac{L - b^2}{b^2 c^2} \right] \Gamma''.$$

We will now put

$$\mathfrak{K} = \frac{KL}{b^2} \qquad \mathfrak{L} = \frac{L - b^2}{c^2 L}.$$

In consequence, the general expression for R_0, S_0, or W_0 will take the form

$$\frac{\mathfrak{K}}{(G + G'')^{\frac{3}{2}}} \left[f + (1 + \mathfrak{L}) \Gamma' + (2 - \mathfrak{L}) \Gamma'' \right].$$

If we put

$$N = \frac{ar^3 \mathfrak{K}}{(G + G'')^{\frac{3}{2}}} \qquad N' = \frac{N(1 + \mathfrak{L})}{b^2c^2(G + G'')^2} \qquad N'' = \frac{N(2 - \mathfrak{L})}{c^2(G + G'')^2}$$

and substitute for Γ' and Γ'' their values, this expression becomes

$$(N' - N'')\frac{F}{ar^3} + (N'G' + N''G'')\frac{J}{ar^3} + (N + N G'^2 - N'' G''^2)\frac{f}{ar^3}.$$

This can be rendered more suitable for computation by putting

$$P = N' - N'' = \frac{N[-2b^2 + 1 + (1 + b^2)\mathfrak{L}]}{b^2c^2(G + G'')^2}$$
$$Q = N'(G' + G'') = \frac{N(1 + \mathfrak{L})}{b^2(G + G'')}$$
$$V = Q - PG''.$$

Then the expression takes the form

$$P\frac{F}{ar^3} + V\frac{J}{ar^3} + (N + QG' - VG'')\frac{f}{ar^3}.$$

If we call $\frac{F}{ar^3}$, $\frac{J}{ar^3}$, and $\frac{f}{ar^3}$ severally in the cases of R_0, S_0, and W_0 by F_1, J_1, f_1, F_2, J_2, f_2, F_3, J_3, f_3, remembering that $F_1 = 0$, $f_1 = -1$, $f_2 = 0$, and $f_3 = 0$, we shall have

$$R_0 = -(N + QG' - VG'') + VJ_1$$
$$S_0 = PF_2 + VJ_2$$
$$W_0 = PF_3 + VJ_3.$$

It now only remains to show how the elliptic integrals K and L may be computed. If we adopt a new variable, T^0, such that

$$\sin(2T - T^0) = c^0 \sin T^0$$

where $c^0 = \frac{1 - b}{1 + b}$, we shall have the following equations:

$$\cos(2T - T^0) = \sqrt{(1 - c^{0\,2} \sin^2 T^0)} = \Delta$$
$$\cos 2T = \Delta \cos T^0 - c^0 \sin^2 T^0$$
$$\sin 2T = \Delta \sin T^0 + c^0 \sin T^0 \cos T^0$$
$$= \sin T^0 (c^0 \cos T^0 + \Delta)$$
$$2dT = \frac{dT^0}{\Delta}(c^0 \cos T^0 + \Delta)$$
$$\sqrt{(1 - c^2 \sin^2 T)} = \frac{c^0 \cos T^0 + \Delta}{1 + c^0}$$
$$\frac{dT}{\sqrt{(1 - c^2 \sin^2 T)}} = \frac{1 + c^0}{2}\frac{dT^0}{\Delta}$$

which constitute the well-known transformation of Landen. It is plain, from the values of $\sin(2T - T^0)$ and $\cos(2T - T^0)$ that, when T passes from the value 0 to the value $\frac{\pi}{2}$, T^0 passes from 0 to π. Hence,

$$\int_0^{\frac{\pi}{2}} \frac{dT}{\sqrt{(1 - c^2 \sin^2 T)}} = (1 + c^0) \int_0^{\frac{\pi}{2}} \frac{dT^0}{\sqrt{(1 - c^{02} \sin^2 T^0)}}$$

or

$$F^1(c) = (1 + c^0)\, F^1(c^0).$$

If we take c^{00} the same function of c^0 that c^0 is of c, and, again, in like manner, derive c^{000}, and so on, the quantities c, c^0, c^{00}, etc., diminish, and, as $F^1(0) = \frac{\pi}{2}$, we shall have

$$F^1(c) = \frac{\pi}{2}(1 + c^0)(1 + c^{00})(1 + c^{000}) \ldots$$

If the *moduli* complementary to c^0, c^{00}, etc., are denoted by b^0, b^{00}, etc. we shall have $b^0 = \sqrt{1 - c^{02}}$ and $b = \frac{1 - c^0}{1 + c^0}$. Consequently,

$$(1 + c^0) = \frac{b^0}{\sqrt{b}}.$$

Hence,

$$K = \sqrt{\frac{b^0 b^{00} b^{000} \ldots}{b}}.$$

From the equations

$$\frac{dT}{\sqrt{(1 - c^2 \sin^2 T)}} = \frac{1 + c^0}{2} \frac{dT^0}{\Delta} \qquad \sin^2 T = \tfrac{1}{2}(1 + c^0 \sin^2 T^0 - \Delta \cos T^0)$$

we obtain

$$\int_0^{\frac{\pi}{2}} \frac{A + B \sin^2 T}{\sqrt{(1 - c^2 \sin^2 T)}} dT = (1 + c^0) \int_0^{\frac{\pi}{2}} \frac{A + \frac{B}{2} + B \frac{c^0}{2} \sin^2 T^0}{\Delta} dT^0.$$

If this process of transformation is continued as in the case of the former integral we find that

$$\int_0^{\frac{\pi}{2}} \frac{A + B \sin^2 T}{\sqrt{(1 - c^2 \sin^2 T)}} dT = \frac{\pi}{2} K \left[A + \frac{B}{2}\left(1 + \frac{c^0}{2} + \frac{c^0 c^{00}}{4} + \frac{c^0 c^{00} c^{000}}{8} + \ldots\right)\right].$$

In the case of $E^1(c)$ we have $A = 1$ and $B = -c^2$; hence,

$$L = 1 - \frac{c^2}{2} - \frac{c^2 c^0}{4} - \frac{c^2 c^0 c^{00}}{8} - \ldots$$

As we have

$$1 - \frac{c^2}{2} - \frac{c^2 c^0}{4} = \frac{c^2}{4c^0} = \frac{b}{b^{0\,2}}$$

and as we may, for our purpose, cut off the series at the term which contains c^{000}, and with sufficient approximation put

$$1 + \tfrac{1}{2} c^{000} = \sqrt{1 + c^{000}} = \sqrt{\frac{2\sqrt{c^{000}}}{c^{00}}} = \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}$$

we may put

$$L = \frac{b}{b^{0\,2}}\left[1 - \tfrac{1}{2} c^{0\,2} c^{00} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right].$$

In like manner

$$\frac{L - b^2}{c^2} = \tfrac{1}{2}\left[1 - \frac{c^0}{2} - \frac{c^0 c^{00}}{4} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right]$$

$$\mathfrak{K} = \sqrt{\frac{b^{00} b^{000}}{b^2 b^{0\,2}}}\left[1 - \tfrac{1}{2} c^{0\,2} c^{00} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right]$$

$$\frac{(1 + b^2)\mathfrak{L} - 2b^2 + 1}{b^2 c^2} = \mathfrak{L}' = \frac{2 - c^2 - \frac{(1 - c^2 + c^4) b^{0\,2}}{8b}\left(1 + \tfrac{1}{2} c^{00} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right)}{\frac{b^2}{b^{0\,2}}\left[1 - \tfrac{1}{2} c^{0\,2} c^{00} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right]}$$

$$\frac{1 + \mathfrak{L}}{b^2} = \mathfrak{N} = \frac{\tfrac{1}{2} - \tfrac{1}{2} c^2 - \frac{1 + c^2}{2}\left[\frac{c^0}{2} + \frac{c^0 c^{00}}{4} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right]}{\frac{b^2}{b^{0\,2}}\left[1 - \tfrac{1}{2} c^{0\,2} c^{00} \frac{\sqrt{b^{000}}}{\sqrt[4]{b^{00}}}\right]}.$$

The common logarithms of the last three functions are tabulated at the end of this memoir. In order to make the data of Legendre's Tables in the second volume of his *Théorie des Fonctions Elliptiques* available, c has been put $= \sin\theta$, and θ adopted as the argument. The quantities are given to eight places of decimals, having been computed with ten. They are tabulated at intervals of a tenth of a degree, and are given from $\theta = 0$ up to $\theta = 50°$. Beyond the latter limit they will scarcely be needed and the interpolation of the tables becomes difficult. Should values, beyond the limit of the table, be wanted, it will be easier to compute them directly from the formulæ than to derive them by interpolation from values tabulated at intervals of 0°. 1 in the value of θ.

Recapitulation of the formulæ needed for the application of this method.

For the benefit of those who wish to make a numerical application of this method, I have here gathered together and arranged, in proper order, all the formulæ necessary to be used. For the signification of the symbols, the preceding discussion must be consulted.

Compute the constants I, Π, Π', k, K, k', K', and C, which are functions of the elements of the two orbits, by means of the equations

$$\begin{aligned}
\sin I\cos(\Pi-\omega) &= -\sin i\cos i' + \cos i\sin i'\cos(\Omega'-\Omega)\\
\sin I\sin(\Pi-\omega) &= -\sin i'\sin(\Omega'-\Omega)\\
\sin I\cos(\Pi'-\omega') &= \cos i\sin i' - \sin i\cos i'\cos(\Omega'-\Omega)\\
\sin I\sin(\Pi'-\omega') &= -\sin i\sin(\Omega'-\Omega)\\
k\cos(K-\Pi) &= \cos\Pi'\\
k\sin(K-\Pi) &= -\cos I\sin\Pi'\\
k'\cos(K'-\Pi) &= \cos I\cos\Pi'\\
k'\sin(K'-\Pi) &= -\sin\Pi'\\
C &= a'^2e'^2.
\end{aligned}$$

The circumference, with reference to the variable E, will now be divided into a certain number of equal parts, which number ought to be a multiple of 4, and should be large or small as the perturbations are more or less irregular through the variation of the distance of the two planets. For each of these values of E, the values of the varying quantities in the left members of the following equations must be calculated. Here a useful check against large errors may be had by adding the first, third, fifth, etc., numerical values of any one of these quantities, and again the second, fourth, sixth, etc. The difference of the two sums should be very small, except in case of certain angles, where one sum may exceed the other by nearly 180°. The same test may be applied to the logarithms of a quantity, provided it does not change sign and does not approach zero very closely.

$$\begin{aligned}
r\cos v &= a(\cos E - e)\\
r\sin v &= a\cos\varphi\sin E\\
A &= r^2 + 2ka'e'r\cos(v+K) + a'^2\\
B\cos\epsilon &= ka'r\cos(v+K) + a'^2e'\\
B\sin\epsilon &= k'a'\cos\varphi'.\, r\sin(v+K')\\
g &= B^2C\sin^2\epsilon\\
h &= \tfrac{1}{2}\left[A - C + \sqrt{(A+C)^2 - 4B^2}\right]\\
l &= \tfrac{1}{2}\left[A - C - \sqrt{(A+C)^2 - 4B^2}\right].
\end{aligned}$$

Find G, G', and G'' by trial from the equations

$$\begin{aligned}
G &= h - \frac{g}{G(G-l)}\\
G' &= l + \frac{g}{G'(h-G')}\\
G'' &= \frac{g}{(h+G'')(l+G'')}.
\end{aligned}$$

Approximate values are

$$
\begin{aligned}
G &= h - \frac{g}{h\,(h-l)} \\
G' &= l + \frac{g}{l\,(h-l)} \\
G'' &= \frac{g}{\left(h + \frac{g}{hl}\right)\left(l + \frac{g}{hl}\right)} \\
\sin^2\theta &= \frac{G' + G''}{G + G''}.
\end{aligned}
$$

From the tables at the end of this memoir, with the argument θ, take out the values of $\log \mathfrak{K}$, $\log \mathfrak{L}'$, and $\log \mathfrak{N}$.

$$
\begin{aligned}
N &= \frac{a r_2 \mathfrak{K}}{(G+G'')^{\frac{3}{2}}} \\
P &= \frac{N \mathfrak{L}'}{(G+G'')^2} \\
Q &= \frac{N \mathfrak{N}}{G+G''} \\
V &= Q - PG'' \\
J_1 &= a'^2 \cos^2\varphi' \,[1 - \sin^2 I \sin^2 (v + \Pi)] + G'' \\
J_2 &= k a' e' r \sin (v + K) - \tfrac{1}{2} a'^2 \cos^2 \varphi' \sin^2 I \sin 2\,(v + \Pi) \\
J_3 &= \frac{a'^2}{a} \cos^2\varphi' \sin I \cos I . r \sin (v + \Pi) - \frac{a'}{a} e' \sin I \sin \Pi' . r^2 \\
F_2 &= - a'^2 \sin\varphi' \cos\varphi' \cos I . B \sin\epsilon \\
F_3 &= - \frac{a'^2}{a} \sin\varphi' \cos\varphi' \sin I . r \cos (v + \Pi) . B \sin \epsilon \\
R_0 &= - N - QG' + VJ_1 \\
S_0 &= PF_2 + VJ_2 \\
W_0 &= PF_3 + VJ_3.
\end{aligned}
$$

The secular variations of the elements will be given by the following equations:

$$
\begin{aligned}
\left[\frac{de}{dt}\right]_{00} &= \frac{m'n}{1+m} \cos\varphi . M_E \Big[\sin v . R_0 + (\cos v + \cos E)\, S_0 \Big] \\
e \left[\frac{d\chi}{dt}\right]_{00} &= \frac{m'n}{1+m} \cos\varphi . M_E \Big[- \cos v . R_0 + \left(\frac{r}{a \cos^2\varphi} + 1\right) \sin v . S_0 \Big] \\
\left[\frac{di}{dt}\right]_{00} &= \frac{m'n}{1+m} \sec\varphi . M_E \Big[\cos u . W_0 \Big] \\
\sin i \left[\frac{d\Omega}{dt}\right]_{00} &= \frac{m'n}{1+m} \sec\varphi . M_E \Big[\sin u . W_0 \Big] \\
\left[\frac{d\pi}{dt}\right]_{00} &= \left[\frac{d\chi}{dt}\right]_{00} + 2 \sin^2 \frac{i}{2} . \left[\frac{d\Omega}{dt}\right]_{00} \\
\left[\frac{dL}{dt}\right]_{00} &= \frac{m'n}{1+m} M_E \Big[-2 \frac{r}{a} R_0 \Big] + 2 \sin^2 \frac{\varphi}{2} . \left[\frac{d\chi}{dt}\right]_{00} + 2 \sin^2 \frac{i}{2} . \left[\frac{d\Omega}{dt}\right]_{00}.
\end{aligned}
$$

EXAMPLE.

Computation of the Secular Perturbations of Mercury produced by the Action of Venus.

The elements of the two planets, adopted for the epoch 1850.0, are

Mercury.	Venus.
$n = 5381016''.26$	$n' = 2106641''.357$
$e = 0.20560476$	$e' = 0.00684311$
$\pi = 75^\circ\ 7'\ 13''.62$	$\pi' = 129^\circ\ 27'\ 42''.83$
$i = 7^\circ\ 0'\ 7''.71$	$i' = 3^\circ\ 23'\ 35''.01$
$\Omega = 46^\circ\ 33'\ 8''.63$	$\Omega' = 75^\circ\ 19'\ 53''.08$
$\log a = 9.5878217$	$\log a' = 9.8593378$
$m = \frac{1}{5000000}$	

From these are deduced

$I = 4^\circ\ 20'\ 42''.98$	$K = 305^\circ\ 43'\ 2''.46$	$\log k = 9.9999176$
$\Pi = 230^\circ\ 39'\ 31''.39$	$K' = 305^\circ\ 47'\ 57''.54$	$\log C = 5.3891826$
$\Pi' = 284^\circ\ 54'\ 1''.18$	$\log k = 9.9988328$	$C = 0.00002450$

The circumference is now divided into twelve parts with respect to E, the eccentric anomaly of Mercury. The values of the various quantities employed in the computation, computed for each of the points of division, are tabulated below. The result of the application of the test, mentioned above, is given at the foot of the column, opposite to the symbols S and S', whenever it is supposed to be useful. The numbers given are affected with asterisks when the additions have been made on the numbers which correspond to the logarithms in the column of values.

E	log. r	v			A	log. B	ϵ			log. g
°		°	′	″			°	′	″	
0	9.4878584	0	0	0.00	0.61954395	9.3505444	306	25	17.64	3.90151
30	9.5026623	36	32	7.50	0.62743501	9.3671640	342	33	14.83	3.07719
60	9.5407098	70	50	41.41	0.64711632	9.4050438	16	26	41.01	3.10312
90	9.5878217	101	51	53.65	0.67563289	9.4506321	47	9	9.28	4.02085
120	9.6303194	129	46	44.60	0.70650301	9.4909308	74	53	39.98	4.34050
150	9.6589887	155	27	29.02	0.73029576	9.5171866	100	32	23.25	4.40878
180	9.6690267	180	0	0.00	0.73831733	9.5249278	125	10	50.07	4.26384
210	9.6589887	204	32	30.98	0.72725905	9.5130385	149	56	52.18	3.81457
240	9.6303194	230	13	15.40	0.70124328	9.4833852	175	57	47.29	2.05108
270	9.5878217	258	8	6.35	0.66955948	9.4412922	204	16	31.00	3.49971
300	9.5407098	289	9	18.59	0.64185659	9.3963533	235	38	26.28	4.01534
330	9.5026623	323	27	52.50	0.62439830	9.3618721	270	4	31.93	4.11293
S . .	. . . .	. .	. .	. .	4.05458048	6.6511853	934	32	42.27	. . . .
S' . .	. . . .	. .	. .	. .	4.05458049	6.6511855	1114	32	42.47	. . . .

E	h	l	G	G'	G''	θ	log. $\mathfrak{K}$
°						° ′ ″	
0	0.52358611	0.09593335	0.52358255	0.09595277	0.00001587	25 20 53.91	0.0667815
30	0.52390824	0.10350226	0.52390770	0.10350501	0.00000220	26 23 25.40	0.0726785
60	0.52384405	0.12324776	0.52384345	0.12325033	0.00000196	29 0 59.16	0.0888373
90	0.52344857	0.15215982	0.52344317	0.15217839	0.00001317	32 37 46.67	0.1142938
120	0.52319735	0.18328117	0.52318503	0.18331632	0.00002284	36 17 45.75	0.1442958
150	0.52358284	0.20668842	0.52356739	0.20672755	0.00002368	38 55 52.65	0.1687224
180	0.52446108	0.21383175	0.52444981	0.21385939	0.00001617	39 41 12.28	0.1762011
210	0.52500793	0.20222662	0.52500408	0.20223662	0.00000615	38 21 51.22	0.1632515
240	0.52470763	0.17651115	0.52470757	0.17651133	0.00000012	35 27 1.86	0.1369807
270	0.52391066	0.14562431	0.52390907	0.14563005	0.00000414	31 49 7.12	0.1082397
300	0.52329644	0.11853565	0.52329155	0.11855724	0.00001670	28 25 30.44	0.0850327
330	0.52323371	0.10114009	0.52322784	0.10117046	0.00002450	26 5 20.74	0.0709442
S - -	3.14309266	0.91134083	3.14305996	0.91144738	0.00007386	194 13 23.40	0.6981291
S′ - -	3.14309195	0.91134152	3.14305925	0.91144808	0.00007384	194 13 23.80	0.6981301

E	log. $\mathfrak{L}'$	log. $\mathfrak{M}$	log. N	log. P	log. Q	log. V	log. J_1
°							
0	0.3610703	0.2748567	9.0518226	9.9748963	9.6076810	9.6076649	9.7171747
30	0.3687562	0.2834450	9.0869399	0.0171829	9.6511283	9.6511261	9.7161627
60	0.3897436	0.3068691	9.1792740	0.1306114	9.7669400	9.7669380	9.7168407
90	0.4225948	0.3434556	9.2994382	0.2842720	9.9240133	9.9240002	9.7181351
120	0.4609870	0.3860837	9.4147450	0.4383836	0.0821545	0.0821320	9.7186740
150	0.4919942	0.4204077	9.4960332	0.5500430	0.1974487	0.1974255	9.7181915
180	0.5014421	0.4308492	9.5225000	0.5845071	0.2336317	0.2336158	9.7171751
210	0.4850679	0.4127493	9.4887989	0.5335312	0.1813804	0.1813744	9.7163236
240	0.4516579	0.3757381	9.4055651	0.4173882	0.0613858	0.0613857	9.7162443
270	0.4148054	0.3347895	9.2928157	0.2691023	9.9083458	9.9083417	9.7171416
300	0.3848118	0.3013686	9.1761378	0.1234346	9.7587489	9.7587321	9.7183721
330	0.3664971	0.2809213	9.0860239	0.0150988	9.6482341	9.6482093	9.7185270
S - -	2.5497127	2.0757654	5.7500445	1.6692212	9.5105419	9.5104685	8.3044809
S′ - -	2.5497156	2.0757684	5.7500498	1.6692302	9.5105506	9.5104772	8.3044815

E	log. J_2	log. J_3	log. F_2	log. F_3	log. R_0	log. S_0	log. W_0
°							
0	n7.4321671	n8.3837285	6.8088312	n5.3916432	8.7760911	n6.6886872	n7.9924224
30	n6.7963083	n8.5099324	6.3966713	n3.8820117	8.8092004	n5.3190515	n8.1610823
60	7.2616976	n8.4788955	n6.4096375	n4.9613828	8.9109724	6.8580694	n8.2461381
90	7.4216280	n8.2575909	n6.8685040	n5.6972542	9.0478487	6.9002047	n8.1843223
120	7.3047658	6.7021384	n7.0283282	n5.9515414	9.1783301	n6.6917105	6.5600086
150	7.0091948	8.3158080	n7.0624655	n5.9675881	9.2656128	n7.3958789	8.5088240
180	6.5998867	8.5688794	n6.9899995	n5.7539798	9.2869000	n7.4874988	8.8010010
210	6.5740806	8.6552729	n6.7653615	n5.1245368	9.2427427	n7.1525123	8.8363593
240	6.8487789	8.6332314	n5.8836161	4.0827215	9.1508864	6.7875671	8.6946448
270	6.8412620	8.4906201	6.6079307	n5.2855935	9.0333867	7.1190054	8.3983398
300	n6.6329728	8.0916691	6.8657465	n5.6718326	8.9135270	6.8627036	7.8465591
330	n7.3581667	n7.8939066	6.9145405	n5.6967794	8.8179243	n6.2157912	n7.5484891
S - -	- - - -	- - - -	- - - -	- - - -	4.2167070	−0.001989228*	+0.09268013*
S′ - -	- - - -	- - - -	- - - -	- - - -	4.2167156	−0.001984156*	+0.09258717*

E	$R_0 \sin v + S_0(\cos v + \cos E)$	$-R_0 \cos v + S_0\left(\frac{r}{a\cos^2\phi}+1\right)\sin v$	$W_0 \cos u$	$W_0 \sin u$	$-2\frac{r}{a}R_0$
°					
0	− 0.00097660	− 0.0597161	− 0.00863059	− 0.00469931	− 0.0948763
30	+ 0.03833155	− 0.0518053	− 0.00610021	− 0.01314386	− 0.1059427
60	+ 0.07755206	− 0.0254115	+ 0.00288259	− 0.01738805	− 0.1461808
90	+ 0.10909871	+ 0.0245450	+ 0.00991450	− 0.01163594	− 0.2232948
120	+ 0.11643388	+ 0.0956574	− 0.00033746	+ 0.00013397	− 0.3325506
150	+ 0.08098430	+ 0.1653789	− 0.03219222	− 0.00226584	− 0.4343200
180	+ 0.00614510	+ 0.1935976	− 0.05554168	− 0.03024215	− 0.4668045
210	− 0.07011566	+ 0.1603978	− 0.04118259	− 0.05487000	− 0.4120403
240	− 0.10947664	+ 0.0895491	− 0.00962480	− 0.04855987	− 0.3121865
270	− 0.10595401	+ 0.0195723	+ 0.00719195	− 0.02396722	− 0.2159816
300	− 0.07680512	− 0.0282224	+ 0.00519678	− 0.00472486	− 0.1470433
330	− 0.03941924	− 0.0526511	− 0.00350168	+ 0.00049010	− 0.1080923
S . .	+ 0.01287268	+ 0.2654541	− 0.06605516	− 0.10548027	− 1.4996420
S′ . .	+ 0.01292565	+ 0.2654376	− 0.06587025	− 0.10539276	− 1.4996717
	+ 0.02579833	+ 0.5308917	− 0.13192541	− 0.21087303	− 2.9993137

Dividing the numbers at the foot of the last five columns by 12, we have the average values of the several functions written at the top. And, leaving the mass of Venus indefinite, we have

		log. coeff.
$\left[\frac{de}{dt}\right]_{00} = +$	$11321''.28\ m'$	4.0538954
$\left[\frac{d\chi}{dt}\right]_{00} = +$	$1133122''\ m'$	6.0542766
$\left[\frac{di}{dt}\right]_{00} = -$	$60449''.22\ m'$	n 4.7813907
$\left[\frac{d\Omega}{dt}\right]_{00} = -$	$792604''.4\ m'$	n 5.8990565
$\left[\frac{d\pi}{dt}\right]_{00} = +$	$1127210''\ m'$	6.0520049
$\left[\frac{dL}{dt}\right]_{00} = -$	$1326648''.7\ m'$	n 6.1227559.

The eccentricity e is supposed to be expressed in seconds of arc; if the variation in parts of the radius is wanted, the result given above must be multiplied by the factor whose logarithm is 94.6855749. It is scarcely necessary to add that the unit of time is the Julian year, and that m' must be expressed in parts of the sun's mass.

If we adopt Leverrier's value of m', viz., $m' = \frac{1}{401847}$, we have the values of the secular variations given below. Alongside, for the sake of comparison, I put Leverrier's values, deduced from the series expanded in

powers of the eccentricities and mutual inclination of the planes of the orbits. (*Annales de l'Observatoire de Paris. Mémoires.* Tome V, pp. 6-7-21.)

	Leverrier's Values
$\left[\frac{de}{dt}\right]_{00} = +\ 0''.0281731$	$+\ 0''.02823$
$\left[\frac{d\pi}{dt}\right]_{00} = +\ 2''.805073$	$+\ 2''.8064$
$\left[\frac{di}{dt}\right]_{00} = -\ 0''.1504284$	$-\ 0''.15044$
$\left[\frac{d\Omega}{dt}\right]_{00} = -\ 1''.972403$	$-\ 1''.9702$
$\left[\frac{dL}{dt}\right]_{00} = -\ 3''.301377$	$-\ 3''.3282$.

Table of the Values of Three Elliptic Integrals employed in this Memoir.

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
°									
0.0	0.00000000			0.27300127			0.17609126		
		+ 99			+ 132			+ 149	
0.1	00000099		+199	27300259		+265	17609275		+297
		298			397			446	
0.2	00000397		198	27300656		265	17609721		298
		496			662			744	
0.3	00000893		199	27301318		264	17610465		298
		695			926			1042	
0.4	00001588		198	27302244		265	17611507		298
		893			1191			1340	
0.5	0.00002481		+198	0.27303435		+264	0.17612847		+297
		+1091			+ 1455			+ 1637	
0.6	00003572		199	27304890		265	17614484		298
		1290			1720			1935	
0.7	00004862		198	27306610		264	17616419		297
		1488			1984			2232	
0.8	00006350		199	27308594		265	17618651		299
		1687			2249			2531	
0.9	00008037		199	27310843		265	17621182		297
		1886			2514			2828	
1.0	0.00009923		+198	0.27313357		+265	0.17624010		+297
		+2084			+ 2779			+ 3125	
1.1	00012007		198	27316136		264	17627135		299
		2282			3043			3424	
1.2	00014289		199	27319179		265	17630559		297
		2481			3308			3721	
1.3	00016770		199	27322487		264	17634280		298
		2680			3572			4019	
1.4	00019450		198	27326059		266	17638299		298
		2878			3838			4317	
1.5	0.00022328		+199	0.27329897		+264	0.17642616		+298
		+3077			+ 4102			+ 4615	
1.6	00025405		198	27333999		265	17647231		298
		3275			4367			4913	
1.7	00028680		200	27338366		265	17652144		298
		3475			4632			5211	
1.8	00032155		198	27342998		264	17657355		297
		3673			4896			5508	
1.9	00035828		198	27347894		266	17662863		299
		3871			5162			5807	
2.0	0.00039699		+200	0.27353056		+264	0.17668670		+297
		+4071			+ 5426			+ 6104	
2.1	00043770		198	27358482		266	17674774		299
		4269			5692			6403	
2.2	00048039		199	27364174		264	17681177		297
		4468			5956			6700	
2.3	00052507		199	27370130		266	17687877		299
		4667			6222			6999	
2.4	00057174		199	27376352		264	17694876		298
		4866			6486			7297	
2.5	0.00062040		+199	0.27382838		+266	0.17702173		+298

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
°									
2.5	0.00062040	+5065	+199	0.27382838	+ 6752	+266	0.17702173	+ 7595	+298
2.6	00067105	5263	198	27389590	7017	265	17709768	7894	299
2.7	00072368	5463	200	27396607	7282	265	17717662	8191	297
2.8	00077831	5662	199	27403889	7547	265	17725853	8490	299
2.9	00083493	5861	199	27411436	7812	265	17734343	8789	299
3.0	0.00089354	+6061	+200	0.27419248	+ 8078	+266	0.17743132	+ 9087	+298
3.1	00095415	6259	198	27427326	8344	266	17752219	9385	298
3.2	00101674	6459	200	27435670	8608	264	17761604	9684	299
3.3	00108133	6658	199	27444278	8875	267	17771288	9983	299
3.4	00114791	6858	200	27453153	9140	265	17781271	10281	298
3.5	0.00121649	+7057	+199	0.27462293	+ 9405	+265	0.17791552	+10580	+299
3.6	00128706	7256	199	27471698	9671	266	17802132	10879	299
3.7	00135962	7457	201	27481369	9937	266	17813011	11177	298
3.8	00143419	7655	198	27491306	10203	266	17824188	11477	300
3.9	00151074	7856	201	27501509	10468	265	17835665	11775	298
4.0	0.00158930	+8055	+199	0.27511977	+10735	+267	0.17847440	+12075	+300
4.1	00166985	8255	200	27522712	11000	265	17859515	12373	298
4.2	00175240	8455	200	27533712	11267	267	17871888	12673	300
4.3	00183695	8655	200	27544979	11533	266	17884561	12972	299
4.4	00192350	8856	201	27556512	11799	266	17897533	13272	300
4.5	0.00201206	+ 9055	+199	0.27568311	+12065	+266	0.17910805	+13571	+299
4.6	00210261	9255	200	27580376	12332	267	17924376	13870	299
4.7	00219516	9456	201	27592708	12598	266	17938246	14170	300
4.8	00228972	9656	200	27605306	12864	266	17952416	14470	300
4.9	00238628	9856	200	27618170	13132	268	17966886	14769	299
5.0	0.00248484	+10058	+202	0.27631302	+13398	+266	0.17981655	+15070	+301
5.1	00258542	10257	199	27644700	13665	267	17996725	15369	299
5.2	00268799	10459	202	27658365	13931	266	18012094	15669	300
5.3	00279258	10659	200	27672296	14199	268	18027763	15969	300
5.4	00289917	10860	201	27686495	14466	267	18043732	16270	301
5.5	0.00300777	+11061	+201	0.27700961	+14732	+266	0.18060002	+16570	+300
5.6	00311838	11262	201	27715693	15001	269	18076572	16870	300
5.7	00323100	11463	201	27730694	15267	266	18093442	17171	301
5.8	00334563	11665	202	27745961	15535	268	18110613	17471	300
5.9	00346228	11865	200	27761496	15802	267	18128084	17772	301
6.0	0.00358093	+12068	+203	0.27777298	+16070	+268	0.18145856	+18073	+301
6.1	00370161	12269	201	27793368	16338	268	18163929	18374	301
6.2	00382430	12470	201	27809706	16606	268	18182303	18675	301
6.3	00394900	12672	202	27826312	16874	268	18200978	18976	301
6.4	00407572	12874	202	27843186	17141	267	18219954	19277	301
6.5	0.00420446		+203	0.27860327		+269	0.18239231		+301

θ	Log. K			Log. L'			Log. N		
°									
6.5	0.00420446	+13077	+203	0.27860327	+17410	+269	0.18239231	+19578	+301
6.6	00433523	13278	201	27877737	17679	269	18258809	19880	302
6.7	00446801	13480	202	27895416	17946	267	18278689	20182	302
6.8	00460281	13683	203	27913362	18215	269	18298871	20483	301
6.9	00473964	13885	202	27931577	18484	269	18319354	20786	303
7.0	0.00487849	+14088	+203	0.27950061	+18753	+269	0.18340140	+21087	+301
7.1	00501937	14291	203	27968814	19021	268	18361227	21389	302
7.2	00516228	14493	202	27987835	19291	270	18382616	21691	302
7.3	00530721	14696	203	28007126	19559	268	18404307	21994	303
7.4	00545417	14899	203	28026685	19829	270	18426301	22296	302
7.5	0.00560316	+15103	+204	0.28046514	+20098	+269	0.18448597	+22599	+303
7.6	00575419	15305	202	28066612	20368	270	18471196	22902	303
7.7	00590724	15509	204	28086980	20638	270	18494098	23204	302
7.8	00606233	15713	204	28107618	20907	269	18517302	23507	303
7.9	00621946	15916	203	28128525	21177	270	18540809	23811	304
8.0	0.00637862	+16121	+205	0.28149702	+21447	+270	0.18564620	+24114	+303
8.1	00653983	16324	203	28171149	21717	270	18588734	24417	303
8.2	00670307	16528	204	28192866	21988	271	18613151	24721	304
8.3	00686835	16733	205	28214854	22258	270	18637872	25025	304
8.4	00703568	16937	204	28237112	22529	271	18662897	25328	303
8.5	0.00720505	+17141	+204	0.28259641	+22800	+271	0.18688225	+25633	+305
8.6	00737646	17346	205	28282441	23070	270	18713858	25936	303
8.7	00754992	17551	205	28305511	23342	272	18739794	26241	305
8.8	00772543	17756	205	28328853	23613	271	18766035	26546	305
8.9	00790299	17962	206	28352466	23884	271	18792581	26850	304
9.0	0.00808261	+18166	+204	0.28376350	+24156	+272	0.18819431	+27155	+305
9.1	00826427	18372	206	28400506	24427	271	18846586	27460	305
9.2	00844799	18577	205	28424933	24700	273	18874046	27765	305
9.3	00863376	18784	207	28449633	24971	271	18901811	28070	305
9.4	00882160	18989	205	28474604	25243	272	18929881	28376	306
9.5	0.00901149	+19195	+206	0.28499847	+25516	+273	0.18958257	+28681	+305
9.6	00920344	19402	207	28525363	25789	273	18986938	28987	306
9.7	00939746	19608	206	28551152	26061	272	19015925	29293	306
9.8	00959354	19814	206	28577213	26333	272	19045218	29600	307
9.9	00979168	20022	208	28603546	26607	274	19074818	29906	306
10.0	0.00999190	+20228	+206	0.28630153	+26881	+274	0.19104724	+30212	+306
10.1	01019418	20435	207	28657034	27153	272	19134936	30519	307
10.2	01039853	20643	208	28684187	27427	274	19165455	30825	306
10.3	01060496	20850	207	28711614	27701	274	19196280	31133	308
10.4	01081346	21058	208	28739315	27975	274	19227413	31440	307
10.5	0.01102404		+207	0.28767290		+273	0.19258853		+308

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
°									
10.5	0.01102404	+21265	+207	0.28767290	+28248	+273	0.19258853	+31748	+308
10.6	01123669	21474	209	28795538	28524	276	19290601	32055	307
10.7	01145143	21682	208	28824062	28797	273	19322656	32363	308
10.8	01166825	21890	208	28852859	29072	275	19355019	32671	308
10.9	01188715	22099	209	28881931	29348	276	19387690	32979	308
11.0	0.01210814	+22307	+208	0.28911279	+29622	+274	0.19420669	+33287	+308
11.1	01233121	22517	210	28940901	29897	275	19453956	33596	309
11.2	01255638	22725	208	28970798	30173	276	19487552	33905	309
11.3	01278363	22935	210	29000971	30449	276	19521457	34214	309
11.4	01301298	23145	210	29031420	30724	275	19555671	34524	310
11.5	0.01324443	+23354	+209	0.29062144	+31000	+276	0.19590195	+34832	+308
11.6	01347797	23564	210	29093144	31277	277	19625027	35143	311
11.7	01371361	23775	211	29124421	31553	276	19660170	35452	309
11.8	01395136	23985	210	29155974	31830	277	19695622	35762	310
11.9	01419121	24195	210	29187804	32107	277	19731384	36073	311
12.0	0.01443316	+24406	+211	0.29219911	+32384	+277	0.19767457	+36383	+310
12.1	01467722	24617	211	29252295	32661	277	19803840	36693	310
12.2	01492339	24829	212	29284956	32939	278	19840533	37005	312
12.3	01517168	25040	211	29317895	33216	277	19877538	37316	311
12.4	01542208	25251	211	29351111	33494	278	19914854	37627	311
12.5	0.01567459	+25463	+212	0.29384605	+33773	+279	0.19952481	+37939	+312
12.6	01592922	25676	213	29418378	34051	278	19990420	38251	312
12.7	01618598	25888	212	29452429	34330	279	20028671	38563	312
12.8	01644486	26100	212	29486759	34609	279	20067234	38875	312
12.9	01670586	26313	213	29521368	34887	278	20106109	39188	313
13.0	0.01696899	+26526	+213	0.29556255	+35167	+280	0.20145297	+39500	+312
13.1	01723425	26740	214	29591422	35447	280	20184797	39814	314
13.2	01750165	26953	213	29626869	35727	280	20224611	40127	313
13.3	01777118	27166	213	29662596	36006	279	20264738	40440	313
13.4	01804284	27381	215	29698602	36287	281	20305178	40754	314
13.5	0.01831665	+27595	+214	0.29734889	+36568	+281	0.20345932	+41069	+315
13.6	01859260	27809	214	29771457	36848	280	20387001	41382	313
13.7	01887069	28024	215	29808305	37129	281	20428383	41697	315
13.8	01915093	28239	215	29845434	37411	282	20470080	42012	315
13.9	01943332	28454	215	29882846	37692	281	20512092	42327	315
14.0	0.01971786	+28670	+216	0.29920537	+37974	+282	0.20554419	+42642	+315
14.1	02000456	28885	215	29958511	38256	282	20597061	42958	316
14.2	02029341	29101	216	29996767	38539	283	20640019	43274	316
14.3	02058442	29317	216	30035306	38821	282	20683293	43590	316
14.4	02087759	29534	217	30074127	39104	283	20726883	43906	316
14.5	0.02117293		+217	0.30113231		+283	0.20770789		+316

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
14.5°	0.02117293	+29751	+217	0.30113231	+39387	+283	0.20770789	+44222	+316
14.6	02147044	29968	217	30152618	39670	283	20815011	44540	318
14.7	02177012	30184	216	30192288	39954	284	20859551	44857	317
14.8	02207196	30403	219	30232242	40238	284	20904408	45174	317
14.9	02237599	30620	217	30272480	40522	284	20949582	45493	319
15.0	0.02268219	+30839	+219	0.30313002	+40807	+285	0.20995075	+45810	+317
15.1	02299058	31056	217	30353809	41091	284	21040885	46128	318
15.2	02330114	31276	220	30394900	41377	286	21087013	46447	319
15.3	02361390	31494	218	30436277	41661	284	21133460	46766	319
15.4	02392884	31714	220	30477938	41948	287	21180226	47086	320
15.5	0.02424598	+31933	+219	0.30519886	+42233	+285	0.21227312	+47404	+318
15.6	02456531	32152	219	30562119	42520	287	21274716	47725	321
15.7	02488683	32373	221	30604639	42806	286	21322441	48044	319
15.8	02521056	32593	220	30647445	43092	286	21370485	48365	321
15.9	02553649	32814	221	30690537	43380	288	21418850	48685	320
16.0	0.02586463	+33035	+221	0.30733917	+43667	+287	0.21467535	+49007	+322
16.1	02619498	33256	221	30777584	43955	288	21516542	49328	321
16.2	02652754	33478	222	30821539	44243	288	21565870	49649	321
16.3	02686232	33699	221	30865782	44531	288	21615519	49972	323
16.4	02719931	33922	223	30910313	44820	289	21665491	50293	321
16.5	0.02753853	+34144	+222	0.30955133	+45109	+289	0.21715784	+50616	+323
16.6	02787997	34367	223	31000242	45398	289	21766400	50939	323
16.7	02822364	34590	223	31045640	45687	289	21817339	51262	323
16.8	02856954	34814	224	31091327	45977	290	21868601	51586	324
16.9	02891768	35037	223	31137304	46268	291	21920187	51909	323
17.0	0.02926805	+35261	+224	0.31183572	+46557	+289	0.21972096	+52234	+325
17.1	02962066	35485	224	31230129	46849	292	22024330	52558	324
17.2	02997551	35711	226	31276978	47140	291	22076888	52883	325
17.3	03033262	35935	224	31324118	47431	291	22129771	53208	325
17.4	03069197	36161	226	31371549	47724	293	22182979	53533	325
17.5	0.03105358	+36386	+225	0.31419273	+48015	+291	0.22236512	+53859	+326
17.6	03141744	36612	226	31467288	48308	293	22290371	54186	327
17.7	03178356	36839	227	31515596	48600	292	22344557	54512	326
17.8	03215195	37065	226	31564196	48894	294	22399069	54839	327
17.9	03252260	37292	227	31613090	49187	293	22453908	55166	327
18.0	0.03289552	+37520	+228	0.31662277	+49481	+294	0.22509074	+55493	+327
18.1	03327072	37748	228	31711758	49775	294	22564567	55822	329
18.2	03364820	37975	227	31761533	50070	295	22620389	56149	327
18.3	03402795	38204	229	31811603	50365	295	22676538	56479	330
18.4	03440999	38433	229	31861968	50660	295	22733017	56807	328
18.5	0.03479432		+229	0.31912628		+296	0.22789824		+330

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
18.5°	0.03479432	+38662	+229	0.31912628	+50956	+296	0.22789824	+57137	+330
18.6	03518094	38891	229	31963584	51251	295	22846961	57466	329
18.7	03556985	39121	230	32014835	51548	297	22904427	57796	330
18.8	03596106	39351	230	32066383	51845	297	22962223	58127	331
18.9	03635457	39582	231	32118228	52141	296	23020350	58458	331
19.0	0.03675039	+39813	+231	0.32170369	+52440	+299	0.23078808	+58789	+331
19.1	03714852	40045	232	32222809	52737	297	23137597	59120	331
19.2	03754897	40275	230	32275546	53035	298	23196717	59453	333
19.3	03795172	40508	233	32328581	53333	298	23256170	59785	332
19.4	03835680	40741	233	32381914	53633	300	23315955	60117	332
19.5	0.03876421	+40973	+232	0.32435547	+53932	+299	0.23376072	+60451	+334
19.6	03917394	41207	234	32489479	54232	300	23436523	60784	333
19.7	03958601	41440	233	32543711	54532	300	23497307	61118	334
19.8	04000041	41674	234	32598243	54832	300	23558425	61453	335
19.9	04041715	41908	234	32653075	55134	302	23619878	61787	334
20.0	0.04083623	+42144	+236	0.32708209	+55434	+300	0.23681665	+62122	+335
20.1	04125767	42378	234	32763643	55737	303	23743787	62457	335
20.2	04168145	42614	236	32819380	56038	301	23806244	62794	337
20.3	04210759	42850	236	32875418	56341	303	23869038	63130	336
20.4	04253609	43086	236	32931759	56644	303	23932168	63467	337
20.5	0.04296695	+43323	+237	0.32988403	+56947	+303	0.23995635	+63804	+337
20.6	04340018	43561	238	33045350	57251	304	24059439	64141	337
20.7	04383579	43798	237	33102601	57556	305	24123580	64479	338
20.8	04427377	44036	238	33160157	57859	303	24188059	64818	339
20.9	04471413	44274	238	33218016	58165	306	24252877	65157	339
21.0	0.04515687	+44514	+240	0.33276181	+58470	+305	0.24318034	+65496	+339
21.1	04560201	44752	238	33334651	58776	306	24383530	65835	339
21.2	04604953	44993	241	33393427	59083	307	24449365	66176	341
21.3	04649946	45232	239	33452510	59389	306	24515541	66516	340
21.4	04695178	45474	242	33511899	59696	307	24582057	66858	342
21.5	0.04740652	+45714	+240	0.33571595	+60004	+308	0.24648915	+67198	+340
21.6	04786366	45955	241	33631599	60312	308	24716113	67541	343
21.7	04832321	46198	243	33691911	60620	308	24783654	67883	342
21.8	04878519	46440	242	33752531	60929	309	24851537	68226	343
21.9	04924959	46683	243	33813460	61239	310	24919763	68569	343
22.0	0.04971642	+46926	+243	0.33874699	+61548	+309	0.24988332	+68913	+344
22.1	05018568	47170	244	33936247	61859	311	25057245	69256	343
22.2	05065738	47414	244	33998106	62170	311	25126501	69602	346
22.3	05113152	47658	244	34060276	62481	311	25196103	69947	345
22.4	05160810	47904	246	34122757	62792	311	25266050	70292	345
22.5	0.05208714		+245	0.34185549		+313	0.25336342		+346

θ	Log. K	Δ	Δ²	Log. L′	Δ	Δ²	Log. N	Δ	Δ²
22.5°	0.05208714	+48149	+245	0.34185549	+63105	+313	0.25336342	+70638	+346
22.6	05256863	48396	247	34248654	63417	312	25406980	70985	347
22.7	05305259	48641	245	34312071	63730	313	25477965	71332	347
22.8	05353900	48889	248	34375801	64044	314	25549297	71679	347
22.9	05402789	49136	247	34439845	64358	314	25620976	72027	348
23.0	0.05451925	+49385	+249	0.34504203	+64673	+315	0.25693003	+72375	+348
23.1	05501310	49632	247	34568876	64987	314	25765378	72725	350
23.2	05550942	49882	250	34633863	65303	316	25838103	73074	349
23.3	05600824	50131	249	34699166	65619	316	25911177	73424	350
23.4	05650955	50381	250	34764785	65936	317	25984601	73774	350
23.5	0.05701336	+50631	+250	0.34830721	+66252	+316	0.26058375	+74125	+351
23.6	05751967	50882	251	34896973	66570	318	26132500	74476	351
23.7	05802849	51134	252	34963543	66888	318	26206976	74829	353
23.8	05853983	51385	251	35030431	67207	319	26281805	75181	352
23.9	05905368	51638	253	35097638	67525	318	26356986	75534	353
24.0	0.05957006	+51891	+253	0.35165163	+67845	+320	0.26432520	+75887	+353
24.1	06008897	52144	253	35233008	68166	321	26508407	76241	354
24.2	06061041	52399	255	35301174	68485	319	26584648	76596	355
24.3	06113440	52653	254	35369659	68807	322	26661244	76951	355
24.4	06166093	52907	254	35438466	69129	322	26738195	77306	355
24.5	0.06219000	+53164	+257	0.35507595	+69450	+321	0.26815501	+77662	+356
24.6	06272164	53419	255	35577045	69774	324	26893163	78019	357
24.7	06325583	53676	257	35646819	70096	322	26971182	78377	358
24.8	06379259	53933	257	35716915	70421	325	27049559	78733	356
24.9	06433192	54191	258	35787336	70744	323	27128292	79093	360
25.0	0.06487383	+54449	+258	0.35858080	+71070	+326	0.27207385	+79450	+357
25.1	06541832	54708	259	35929150	71395	325	27286835	79811	361
25.2	06596540	54968	260	36000545	71721	326	27366646	80170	359
25.3	06651508	55227	259	36072266	72048	327	27446816	80531	361
25.4	06706735	55488	261	36144314	72374	326	27527347	80891	360
25.5	0.06762223	+55748	+260	0.36216688	+72702	+328	0.27608238	+81254	+363
25.6	06817971	56011	263	36289390	73031	329	27689492	81615	361
25.7	06873982	56272	261	36362421	73359	328	27771107	81978	363
25.8	06930254	56536	264	36435780	73689	330	27853085	82341	363
25.9	06986790	56798	262	36509469	74019	330	27935426	82706	365
26.0	0.07043588	+57063	+265	0.36583488	+74349	+330	0.28018132	+83070	+364
26.1	07100651	57327	264	36657837	74680	331	28101202	83434	364
26.2	07157978	57592	265	36732517	75012	332	28184636	83801	367
26.3	07215570	57858	266	36807529	75344	332	28268437	84166	365
26.4	07273428	58124	266	36882873	75678	334	28352603	84534	368
26.5	0.07331552		+267	0.36958551		+332	0.28437137		+367

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{R}$		
°									
26.5	0.07331552		+267	0.36958551		+332	0.28437137		+367
		+58391			+76010			+84901	
26.6	07389943		268	37034561		335	28522038		368
		58659			76345			85269	
26.7	07448602		267	37110906		335	28607307		368
		58926			76680			85637	
26.8	07507528		270	37187586		335	28692944		370
		59196			77015			86007	
26.9	07566724		268	37264601		335	28778951		370
		59464			77350			86377	
27.0	0.07626188		+271	0.37341951		+338	0.28865328		+370
		+59735			+77688			+86747	
27.1	07685923		270	37419639		336	28952075		371
		60005			78024			87118	
27.2	07745928		271	37497663		339	29039193		372
		60276			78363			87490	
27.3	07806204		272	37576026		338	29126683		373
		60548			78701			87863	
27.4	07866752		273	37654727		339	29214546		372
		60821			79040			88235	
27.5	0.07927573		+273	0.37733767		+339	0.29302781		+374
		+61094			+79379			+88609	
27.6	07988667		273	37813146		341	29391390		374
		61367			79720			88983	
27.7	08050034		275	37892866		341	29480373		375
		61642			80061			89358	
27.8	08111676		275	37972927		342	29569731		376
		61917			80403			89734	
27.9	08173593		276	38053330		342	29659465		376
		62193			80745			90110	
28.0	0.08235786		+276	0.38134075		+343	0.29749575		+376
		+62469			+81088			+90486	
28.1	08298255		276	38215163		344	29840061		378
		62745			81432			90864	
28.2	08361000		279	38296595		343	29930925		379
		63024			81775			91243	
28.3	08424024		278	38378370		346	30022168		378
		63302			82121			91621	
28.4	08487326		279	38460491		346	30113789		380
		63581			82467			92001	
28.5	0.08550907		+280	0.38542958		+346	0.30205790		+379
		+63861			+82813			+92380	
28.6	08614768		280	38625771		346	30298170		382
		64141			83159			92762	
28.7	08678909		281	38708930		349	30390932		381
		64422			83508			93143	
28.8	08743331		282	38792438		348	30484075		383
		64704			83856			93526	
28.9	08808035		283	38876294		349	30577601		382
		64987			84205			93908	
29.0	0.08873022		+282	0.38960499		+350	0.30671509		+384
		+65269			+84555			+94292	
29.1	08938291		285	39045054		350	30765801		385
		65554			84905			94677	
29.2	09003845		284	39129959		351	30860478		384
		65838			85256			95061	
29.3	09069683		285	39215215		352	30955539		386
		66123			85608			95447	
29.4	09135806		287	39300823		353	31050986		386
		66410			85961			95833	
29.5	0.09202216		+286	0.39386784		+354	0.31146819		+388
		+66696			+86315			+96221	
29.6	09268912		287	39473099		353	31243040		387
		66983			86668			96608	
29.7	09335895		289	39559767		355	31339648		390
		67272			87023			96998	
29.8	09403167		289	39646790		356	31436646		388
		67561			87379			97386	
29.9	09470728		290	39734169		355	31534032		390
		67851			87734			97776	
30.0	0.09538579		+289	0.39821903		+358	0.31631808		+391
		+68140			+88092			+98167	
30.1	09606719		292	39909995		358	31729975		392
		68432			88450			98559	
30.2	09675151		292	39998445		358	31828534		392
		68724			88808			98951	
30.3	09743875		293	40087253		359	31927485		393
		69017			89167			99344	
30.4	09812892		293	40176420		361	32026829		394
		69310			89528			99738	
30.5	0.09882202		+295	0.40265948		+360	0.32126567		+394

θ	Log. K			Log. L′			Log. N		
°									
30.5	0.09882202		+295	0.40265948		+360	0.32126567		+394
		+69605			+89888			+100132	
30.6	09951807		294	40355836		362	32226699		395
		69899			90250			100527	
30.7	10021706		296	40446086		362	32327226		397
		70195			90612			100924	
30.8	10091901		297	40536698		363	32428150		396
		70492			90975			101320	
30.9	10162393		297	40627673		364	32529470		398
		70789			91339			101718	
31.0	0.10233182		+298	0.40719012		+365	0.32631188		+399
		+71087			+91704			+102117	
31.1	10304269		299	40810716		365	32733305		398
		71386			92069			102515	
31.2	10375655		300	40902785		366	32835820		400
		71686			92435			102915	
31.3	10447341		300	40995220		368	32938735		402
		71986			92803			103317	
31.4	10519327		302	41088023		367	33042052		400
		72288			93170			103717	
31.5	0.10591615		+301	0.41181193		+370	0.33145769		+403
		+72589			+93540			+104120	
31.6	10664204		304	41274733		368	33249889		403
		72893			93908			104523	
31.7	10737097		304	41368641		371	33354412		404
		73197			94279			104927	
31.8	10810294		304	41462920		371	33459339		405
		73501			94650			105332	
31.9	10883795		306	41557570		372	33564671		406
		73807			95022			105738	
32.0	0.10957602		+306	0.41652592		+373	0.33670409		+406
		+74113			+95395			+106144	
32.1	11031715		308	41747987		374	33776553		407
		74421			95769			106551	
32.2	11106136		307	41843756		374	33883104		408
		74728			96143			106959	
32.3	11180864		309	41939899		376	33990063		409
		75037			96519			107368	
32.4	11255901		310	42036418		375	34097431		410
		75347			96894			107778	
32.5	0.11331248		+311	0.42133312		+378	0.34205209		+410
		+75658			+97272			+108188	
32.6	11406906		311	42230584		378	34813397		412
		75969			97650			108600	
32.7	11482875		313	42328234		379	34421997		413
		76282			98029			109013	
32.8	11559157		312	42426263		379	34531010		412
		76594			98408			109425	
32.9	11635751		315	42524671		381	34640435		415
		76909			98789			109840	
33.0	0.11712660		+315	0.42623460		+382	0.34750275		+414
		+77224			+99171			+110254	
33.1	11789884		316	42722631		381	34860529		417
		77540			99552			110671	
33.2	11867424		317	42822183		385	34971200		416
		77857			99937			111087	
33.3	11945281		318	42922120		383	35082287		417
		78175			100320			111504	
33.4	12023456		318	43022440		385	35193791		419
		78493			100705			111923	
33.5	0.12101949		+320	0.43123145		+387	0.35305714		+420
		+78813			+101092			+112343	
33.6	12180762		320	43224237		386	35418057		420
		79133			101478			112763	
33.7	12259895		322	43325715		389	35530820		421
		79455			101867			113184	
33.8	12339350		322	43427582		388	35644004		422
		79777			102255			113606	
33.9	12419127		324	43529837		390	35757610		423
		80101			102645			114029	
34.0	0.12499228		+324	0.43632482		+390	0.35871639		+425
		+80425			+103035			+114454	
34.1	12579653		326	43735517		392	35986093		424
		80751			103427			114878	
34.2	12660404		325	43838944		393	36100971		426
		81076			103820			115304	
34.3	12741480		328	43942764		394	36216275		427
		81404			104214			115731	
34.4	12822884		328	44046978		394	36332006		427
		81732			104608			116158	
34.5	0.12904616		+329	0.44151586		+395	0.36448164		+430

θ	Log. K			Log. L′			Log. R		
° 34.5	0.12904616	+82061	+329	0.44151586	+105003	+395	0.36448164	+116588	+430
34.6	12986677	82392	331	44256589	105400	397	36564752	117017	429
34.7	13069069	82722	330	44361989	105797	397	36681769	117447	430
34.8	13151791	83055	333	44467786	106196	399	36799216	117880	433
34.9	13234846	83388	333	44573982	106595	399	36917096	118312	432
35.0	0.13318234	+83722	+334	0.44680577	+106996	+401	0.37035408	+118745	+433
35.1	13401956	84057	335	44787573	107398	402	37154153	119180	435
35.2	13486013	84394	337	44894971	107799	401	37273333	119616	436
35.3	13570407	84731	337	45002770	108204	405	37392949	120052	436
35.4	13655138	85069	338	45110974	108608	404	37513001	120489	437
35.5	0.13740207	+85408	+339	0.45219582	+109013	+405	0.37633490	+120929	+440
35.6	13825615	85749	341	45328595	109420	407	37754419	121368	439
35.7	13911364	86091	342	45438015	109828	408	37875787	121808	440
35.8	13997455	86433	342	45547843	110237	409	37997595	122250	442
35.9	14083888	86776	343	45658080	110646	409	38119845	122693	443
36.0	0.14170664	+87122	+346	0.45768726	+111057	+411	0.38242538	+123137	+444
36.1	14257786	87467	345	45879783	111469	412	38365675	123582	445
36.2	14345253	87814	347	45991252	111883	414	38489257	124027	445
36.3	14433067	88161	347	46103135	112296	413	38613284	124474	447
36.4	14521228	88512	351	46215431	112711	415	38737758	124922	448
36.5	0.14609740	+88861	+349	0.46328142	+113128	+417	0.38862680	+125372	+450
36.6	14698601	89212	351	46441270	113546	418	38988052	125821	449
36.7	14787813	89565	353	46554816	113963	417	39113873	126273	452
36.8	14877378	89919	354	46668779	114384	421	39240146	126725	452
36.9	14967297	90273	354	46783163	114804	420	39366871	127179	454
37.0	0.15057570	+90630	+357	0.46897967	+115226	+422	0.39494050	+127633	+454
37.1	15148200	90986	356	47013193	115650	424	39621683	128088	455
37.2	15239186	91344	358	47128843	116073	423	39749771	128546	458
37.3	15330530	91704	360	47244916	116499	426	39878317	129003	457
37.4	15422234	92065	361	47361415	116925	426	40007320	129463	460
37.5	0.15514299	+92426	+361	0.47478340	+117354	+429	0.40136783	+129923	+460
37.6	15606725	92789	363	47595694	117782	428	40266706	130384	461
37.7	15699514	93153	364	47713476	118212	430	40397090	130846	462
37.8	15792667	93519	366	47831688	118643	431	40527936	131310	464
37.9	15886186	93885	366	47950331	119076	433	40659246	131776	466
38.0	0.15980071	+94253	+368	0.48069407	+119510	+434	0.40791022	+132241	+465
38.1	16074324	94622	369	48188917	119944	434	40923263	132708	467
38.2	16168946	94992	370	48308861	120381	437	41055971	133177	469
38.3	16263938	95364	372	48429242	120818	437	41189148	133646	469
38.4	16359302	95737	373	48550060	121257	439	41322794	134117	471
38.5	0.16455039		+373	0.48671317		+439	0.41456911		+472

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
°									
38.5	0.16455039		+373	0.48671317		+439	0.41456911		+472
		+96110			+121696			+134589	
38.6	16551149		376	48793013		442	41591500		473
		96486			122138			135062	
38.7	16647635		377	48915151		442	41726562		475
		96863			122580			135537	
38.8	16744498		377	49037731		443	41862099		476
		97240			123023			136013	
38.9	16841738		380	49160754		446	41998112		476
		97620			123469			136489	
39.0	0.16939358		+380	0.49284223		+446	0.42134601		+479
		+98000			+123915			+136968	
39.1	17037358		382	49408138		447	42271569		479
		98382			124362			137447	
39.2	17135740		384	49532500		449	42409016		481
		98766			124811			137928	
39.3	17234506		383	49657311		451	42546944		481
		99149			125262			138409	
39.4	17333655		387	49782573		450	42685353		485
		99536			125712			138894	
39.5	0.17433191		+386	0.49908285		+454	0.42824247		+483
		+ 99922			+126166			+139377	
39.6	17533113		390	50034451		453	42963624		487
		100312			126619			139864	
39.7	17633425		389	50161070		457	43103488		487
		100701			127076			140351	
39.8	17734126		391	50288146		456	43243839		488
		101092			127532			140839	
39.9	17835218		393	50415678		458	43384678		490
		101485			127990			141329	
40.0	0.17936703		+394	0.50543668		+460	0.43526007		+491
		+101879			+128450			+141820	
40.1	18038582		396	50672118		460	43667827		493
		102275			128910			142313	
40.2	18140857		396	50801028		463	43810140		494
		102671			129373			142807	
40.3	18243528		399	50930401		464	43952947		494
		103070			129837			143301	
40.4	18346598		399	51060238		465	44096248		498
		103469			130302			143799	
40.5	0.18450067		+402	0.51190540		+467	0.44240047		+497
		+103871			+130769			+144296	
40.6	18553938		402	51321309		467	44384343		500
		104273			131236			144796	
40.7	18658211		404	51452545		470	44529139		500
		104677			131706			145296	
40.8	18762888		406	51584251		471	44674435		503
		105083			132177			145799	
40.9	18867971		407	51716428		472	44820234		503
		105490			132649			146302	
41.0	0.18973461		+408	0.51849077		+474	0.44966536		+505
		+105898			+133123			+146807	
41.1	19079359		411	51982200		475	45113343		506
		106309			133598			147313	
41.2	19185668		410	52115798		477	45260656		508
		106719			134075			147821	
41.3	19292387		414	52249873		478	45408477		510
		107133			134553			148331	
41.4	19399520		415	52384426		480	45556808		510
		107548			135033			148841	
41.5	0.19507068		+416	0.52519459		+482	0.45705649		+512
		+107964			+135515			+149353	
41.6	19615032		417	52654974		482	45855002		514
		108381			135997			149867	
41.7	19723413		420	52790971		484	46004869		515
		108801			136481			150382	
41.8	19832214		420	52927452		486	46155251		517
		109221			136967			150899	
41.9	19941435		423	53064419		488	46306150		517
		109644			137455			151416	
42.0	0.20051079		+423	0.53201874		+488	0.46457566		+521
		+110067			+137943			+151937	
42.1	20161146		427	53339817		492	46609503		520
		110494			138435			152457	
42.2	20271640		426	53478252		491	46761960		524
		110920			138926			152981	
42.3	20382560		430	53617178		494	46914941		523
		111350			139420			153504	
42.4	20493910		429	53756598		495	47068445		526
		111779			139915			154030	
42.5	0.20605689		+433	0.53896513		+498	0.47222475		+528

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{R}$		
°									
42.5	0.20605689	+112212	+433	0.53896513	+140413	+498	0.47222475	+154558	+528
42.6	20717901	112646	434	54036926	140911	498	47377033	155086	528
42.7	20830547	113081	435	54177837	141411	500	47532119	155617	531
42.8	20943628	113518	437	54319248	141913	502	47687736	156149	532
42.9	21057146	113957	439	54461161	142416	503	47843885	156683	534
43.0	0.21171103	+114397	+440	0.54603577	+142922	+506	0.48000568	+157218	+535
43.1	21285500	114840	443	54746499	143429	507	48157786	157755	537
43.2	21400340	115283	443	54889928	143937	508	48315541	158294	539
43.3	21515623	115730	447	55033865	144447	510	48473835	158833	539
43.4	21631353	116176	446	55178312	144960	513	48632668	159376	543
43.5	0.21747529	+116626	+450	0.55323272	+145473	+513	0.48792044	+159919	+543
43.6	21864155	117077	451	55468745	145989	516	48951963	160464	545
43.7	21981232	117529	452	55614734	146505	516	49112427	161011	547
43.8	22098761	117985	456	55761239	147025	520	49273438	161560	549
43.9	22216746	118440	455	55908264	147546	521	49434998	162110	550
44.0	0.22335186	+118899	+459	0.56055810	+148068	+522	0.49597108	+162662	+552
44.1	22454085	119358	459	56203878	148592	524	49759770	163215	553
44.2	22573443	119821	463	56352470	149119	527	49922985	163771	556
44.3	22693264	120284	463	56501589	149646	527	50086756	164329	558
44.4	22813548	120750	466	56651235	150176	530	50251085	164887	558
44.5	0.22934298	+121217	+467	0.56801411	+150708	+532	0.50415972	+165448	+561
44.6	23055515	121687	470	56952119	151242	534	50581420	166011	563
44.7	23177202	122158	471	57103361	151777	535	50747431	166575	564
44.8	23299360	122631	473	57255138	152314	537	50914006	167141	566
44.9	23421991	123106	475	57407452	152853	539	51081147	167710	569
45.0	0.23545097	+123584	+478	0.57560305	+153394	+541	0.51248857	+168279	+569
45.1	23668681	124063	479	57713699	153938	544	51417136	168851	572
45.2	23792744	124543	480	57867637	154482	544	51585987	169424	573
45.3	23917287	125027	484	58022119	155029	547	51755411	170000	576
45.4	24042314	125512	485	58177148	155578	549	51925411	170577	577
45.5	0.24167826	+125999	+487	0.58332726	+156129	+551	0.52095988	+171157	+580
45.6	24293825	126488	489	58488855	156681	552	52267145	171737	580
45.7	24420313	126979	491	58645536	157237	556	52438882	172321	584
45.8	24547292	127473	494	58802773	157793	556	52611203	172906	585
45.9	24674765	127967	494	58960566	158353	560	52784109	173492	586
46.0	0.24802732	+128366	+499	0.59118919	+158913	+560	0.52957601	+174082	+590
46.1	24931198	128965	499	59277832	159476	563	53131683	174673	591
46.2	25060163	129466	501	59437308	160041	565	53306356	175265	592
46.3	25189629	129971	505	59597349	160609	568	53481621	175861	596
46.4	25319600	130476	505	59757958	161177	568	53657482	176457	596
46.5	0.25450076		+508	0.59919135		+572	0.53833939		+600

θ	Log. $\mathfrak{K}$			Log. $\mathfrak{L}'$			Log. $\mathfrak{N}$		
°									
46.5	0.25450076	+130984	+508	0.59919135	+161749	+572	0.53833939	+177057	+600
46.6	25581060	131495	511	60080884	162323	574	54010996	177657	600
46.7	25712555	132008	513	60243207	162899	576	54188653	178261	604
46.8	25844563	132522	514	60406106	163476	577	54366914	178866	605
46.9	25977085	133039	517	60569582	164056	580	54545780	179473	607
47.0	0.26110124	+133558	+519	0.60733638	+164639	+583	0.54725253	+180083	+610
47.1	26243682	134080	522	60898277	165223	584	54905336	180694	611
47.2	26377762	134604	524	61063500	165809	586	55086030	181308	614
47.3	26512366	135130	526	61229309	166399	590	55267338	181924	616
47.4	26647496	135658	528	61395708	166990	591	55449262	182541	617
47.5	0.26783154	+136189	+531	0.61562698	+167583	+593	0.55631803	+183162	+621
47.6	26919343	136723	534	61730281	168179	596	55814965	183785	623
47.7	27056066	137257	534	61898460	168777	598	55998750	184409	624
47.8	27193323	137796	539	62067237	169378	601	56183159	185035	626
47.9	27331119	138336	540	62236615	169981	603	56368194	185665	630
48.0	0.27469455	+138878	+542	0.62406596	+170585	+604	0.56553859	+186297	+632
48.1	27608333	139424	546	62577181	171193	608	56740156	186930	633
48.2	27747757	139972	548	62748374	171803	610	56927086	187565	635
48.3	27887729	140521	549	62920177	172415	612	57114651	188205	640
48.4	28028250	141074	553	63092592	173030	615	57302856	188844	639
48.5	0.28169324	+141629	+555	0.63265622	+173648	+618	0.57491700	+189488	+644
48.6	28310953	142187	558	63439270	174267	619	57681188	190133	645
48.7	28453140	142747	560	63613537	174889	622	57871321	190781	648
48.8	28595887	143310	563	63788426	175513	624	58062102	191431	650
48.9	28739197	143875	565	63963939	176141	628	58253533	192083	652
49.0	0.28883072	+144443	+568	0.64140080	+176771	+630	0.58445616	+192738	+655
49.1	29027515	145013	570	64316851	177403	632	58638354	193396	658
49.2	29172528	145587	574	64494254	178038	635	58831750	194056	660
49.3	29318115	146163	576	64672292	178675	637	59025806	194718	662
49.4	29464278	146741	578	64850967	179316	641	59220524	195383	665
49.5	0.29611019	+147323	+582	0.65030283	+179958	+642	0.59415907	+196050	+667
49.6	29758342	147907	584	65210241	180603	645	59611957	196720	670
49.7	29906249	148493	586	65390844	181252	649	59808677	197393	673
49.8	30054742	149084	591	65572096	181903	651	60006070	198068	675
49.9	30203826	+149675	+591	65753999	+182556	+653	60204138	+198745	+677
50.0	0.30353501			0.65936555			0.60402883		

ADDENDUM.

Since the preceding portion of this memoir was in type it has occurred to me that some of the processes might be modified with advantage.

First, the roots of the equation

$$x[(x-A)(x+C)+B^2]+B^2C\sin^2\epsilon=0$$

can be obtained by the well-known trigonometric method. If we put

$$\begin{aligned}p&=\tfrac{1}{3}(A-C)\\ q^2&=p^2-\tfrac{1}{3}(B^2-AC)\\ r&=\tfrac{1}{2}p(p^2-3q^2)+\tfrac{1}{2}B^2C\sin^2\epsilon\\ \sin\theta=\tau&=\frac{r}{q^3}\end{aligned}$$

and if θ is taken between the limits $\pm 90°$, the three quantities G, G', and G'' are given by the equations

$$\begin{aligned}G&=2q\sin\left(60°-\frac{\theta}{3}\right)+p\\ G'&=2q\sin\frac{\theta}{3}+p\\ G''&=2q\sin\left(60°+\frac{\theta}{3}\right)-p.\end{aligned}$$

From these equations we derive the following:

$$\begin{aligned}G+G''&=2\sqrt{3}\,q\cos\frac{\theta}{3}\\ G'+G''&=2\sqrt{3}\,q\cos\left(60°-\frac{\theta}{3}\right)\\ G-G'&=2\sqrt{3}\,q\cos\left(60°+\frac{\theta}{3}\right).\end{aligned}$$

If these values are substituted in the equations

$$\Gamma'=\frac{F+JG'+fG'^2}{(G'+G'')(G-G')}\qquad\qquad \Gamma''=\frac{-F+JG''-fG''^2}{(G+G'')(G'+G'')}$$

we obtain

$$\Gamma'=\frac{F+Jp+f(p^2+2q^2)+2(J+2fp)q\sin\frac{\theta}{3}-2fq^2\cos\frac{2}{3}\theta}{12q^2\cos\left(60°-\frac{\theta}{3}\right)\cos\left(60°+\frac{\theta}{3}\right)}$$

$$\Gamma''=\frac{-[F+Jp+f(p^2+2q^2)]+2(J+2fp)q\sin\left(60°+\frac{\theta}{3}\right)+2fq^2\cos\left(120°+\frac{2}{3}\theta\right)}{12q^2\cos\frac{\theta}{3}\cos\left(60°-\frac{\theta}{3}\right)}.$$

Or, since we have

$$\cos\theta = 4\cos\frac{\theta}{3}\cos\left(60^\circ - \frac{\theta}{3}\right)\cos\left(60^\circ + \frac{\theta}{3}\right)$$

$$\Gamma' = \frac{[F + Jp + f(p^2 + q^2)]\cos\frac{\theta}{3} + (J + 2fp)\,q\sin\frac{2}{3}\theta}{3\,q^2\cos\theta} - \tfrac{1}{3}f$$

$$\Gamma'' = \frac{-[F + Jp + f(p^2 + q^2)]\cos\left(60^\circ + \frac{\theta}{3}\right) + (J + 2fp)\,q\sin\left(120^\circ + \frac{2}{3}\theta\right)}{3\,q^2\cos\theta} - \tfrac{1}{3}f.$$

From these equations we derive

$$\Gamma' + 2\,\Gamma'' + f = \frac{[F + Jp + f(p^2 + q^2)]\sin\frac{\theta}{3} + (J + 2fp)\,q\cos\frac{2}{3}\theta}{\sqrt{3}\,q^2\cos\theta}$$

$$2\,\Gamma' + \Gamma'' + f = \frac{[F + Jp + f(p^2 + q^2)]\sin\left(60^\circ + \frac{\theta}{3}\right) + (J + 2fp)\,q\cos\left(60^\circ - \frac{2}{3}\theta\right)}{\sqrt{3}\,q^2\cos\theta}.$$

The values of R_0, S_0, and W_0 are given by the integral

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{[\Gamma' + 2\,\Gamma'' + f]\cos^2 T + [2\,\Gamma' + \Gamma'' + f]\sin^2 T}{(2\sqrt{3}\,q)^{\frac{3}{2}}\left[\cos\frac{\theta}{3}\cos^2 T + \cos\left(60^\circ + \frac{\theta}{3}\right)\sin^2 T\right]^{\frac{3}{2}}}\,dT$$

provided we attribute to F, J, and f the values they severally have in each case. Let us put

$$m^2 = \cos\frac{\theta}{3} \qquad\qquad n^2 = \cos\left(60^\circ + \frac{\theta}{3}\right)$$

$$a = \frac{F + Jp + f(p^2 + q^2)}{6\sqrt[4]{12}\,q^{\frac{1}{2}}} \qquad\qquad b = \frac{J + 2fp}{6\sqrt[4]{12}\,q^{\frac{1}{2}}}.$$

Then the integral, just given, takes the form

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\left[a\sin\frac{\theta}{3} + b\cos\frac{2}{3}\theta\right]\cos^2 T + \left[a\sin\left(60^\circ + \frac{\theta}{3}\right) + b\cos\left(60^\circ - \frac{2}{3}\theta\right)\right]\sin^2 T}{\cos\theta\,[m^2\cos^2 T + n^2\sin^2 T]^{\frac{3}{2}}}\,dT.$$

In the second place Gauss's processes for approximating to the values of the integrals may be employed instead of those of Legendre. The equation between definite integrals

$$\int_0^{\frac{\pi}{2}}\frac{dT}{\sqrt{(1 - c^2\sin^2 T)}} = (1 + c')\int_0^{\frac{\pi}{2}}\frac{dT}{\sqrt{(1 - c'^2\sin^2 T)}}$$

may be easily transformed into

$$\int_0^{\frac{\pi}{2}} \frac{dT}{[m^2 \cos^2 T + n^2 \sin^2 T]^{\frac{1}{2}}} = \int_0^{\frac{\pi}{2}} \frac{dT}{[m'^2 \cos^2 T + n'^2 \sin^2 T]^{\frac{1}{2}}}$$

where

$$m' = \tfrac{1}{2}(m + n) \qquad\qquad n' = \sqrt{mn}$$

when we remember that

$$c^2 = \frac{m^2 - n^2}{m^2} \qquad\qquad c^0 = \frac{m - n}{m + n}.$$

If this mode of transformation is continued, and we compute

$$\begin{array}{ll} m'' = \tfrac{1}{2}(m' + n') & n'' = \sqrt{m' n'} \\ m''' = \tfrac{1}{2}(m'' + n'') & n''' = \sqrt{m'' n''} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

the series of quantities, m, m', m'', etc., and n, n', n'', etc., converge very rapidly toward a common limit μ, which Gauss has called the *arithmetico-geometrical mean* between m and n. Then,

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dT}{[m^2 \cos^2 T + n^2 \sin^2 T]^{\frac{1}{2}}} = \frac{1}{\mu}\,.$$

The equation

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{A + B \sin^2 T}{\sqrt{(1 - c^2 \sin^2 T)}}\, dT = K\left[A + \frac{B}{2}\left(1 + \frac{c^0}{2} + \frac{c^0 c^{00}}{4} + \frac{c^0 c^{00} c^{000}}{8} + \ldots\right)\right]$$

on putting

$$A = -\frac{1}{m} \qquad B = \frac{2}{m}$$

is readily transformed into

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 T - \cos^2 T}{[m^2 \cos^2 T + n^2 \sin^2 T]^{\frac{1}{2}}}\, dT = \frac{1}{\mu}\left[\frac{m - n}{2(m + n)} + \frac{m - n}{2\,(m + n)}\frac{m' - n'}{2\,(m' + n')} + \ldots\ldots\right].$$

The series within the brackets may be denoted by ν. It can be transformed as follows:

$$\begin{aligned} \nu &= \frac{m^2 - n^2}{8\, m'^2} + \frac{m^2 - n^2}{8\, m'^2}\,\frac{m'^2 - n'^2}{8\, m''^2} + \frac{m^2 - n^2}{8\, m'^2}\,\frac{m'^2 - n'^2}{8\, m''^2}\,\frac{m''^2 - n''^2}{8\, m'''^2} + \ldots \\ &= \frac{m^2 - n^2}{8\, m'^2} + \frac{m^2 - n^2}{8\, m'^2}\,\frac{(m^2 - n^2)^2}{128\, m'^2\, m''^2} + \frac{m^2 - n^2}{8\, m'^2}\,\frac{(m^2 - n^2)^2}{128\, m'^2\, m''^2}\,\frac{(m'^2 - n'^2)^2}{128\, m''^2\, m'''^2} + \ldots \end{aligned}$$

As this mode of transformation may be continued indefinitely, it is plain, that if we compute the series of quantities

$$\lambda = \tfrac{1}{4}\sqrt{(m^2 - n^2)} \qquad \lambda' = \frac{\lambda^2}{m'} \qquad \lambda'' = \frac{\lambda'^2}{m''} \qquad \lambda''' = \frac{\lambda''^2}{m'''}\cdots\cdots$$

we shall have

$$\nu = \frac{2\,\lambda'^2 + 4\,\lambda''^2 + 8\,\lambda'''^2 + \cdots\cdots}{\lambda^2}.$$

The equation

$$\int_0^{\frac{\pi}{2}} \frac{1 - 2\sin^2 T + c^2 \sin^4 T}{[1 - c^2 \sin^2 T]^{\frac{3}{2}}}\, dT = 0$$

is readily transformed into

$$\int_0^{\frac{\pi}{2}} \frac{m^2 \cos^4 T - n^2 \sin^4 T}{[m^2 \cos^2 T + n^2 \sin^2 T]^{\frac{3}{2}}}\, dT = 0\,.$$

Whence we conclude that

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2 T}{[m^2 \cos^2 T + n^2 \sin^2 T]^{\frac{3}{2}}}\, dT = \frac{1 + \nu}{2\,m^2 \mu}$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2 T}{[m^2 \cos^2 T + n^2 \sin^2 T]^{\frac{3}{2}}}\, dT = \frac{1 - \nu}{2\,n^2 \mu}.$$

Substituting these values in the general integral expression for R_0, S_0, and W_0, we get

$$R_0,\ S_0, \text{ or } W_0 = \frac{a}{\cos\theta}\left[\frac{1 + \nu}{2\,\mu}\tan\frac{\theta}{3} + \frac{1 - \nu}{2\,\mu}\tan\left(60^\circ + \frac{\theta}{3}\right)\right]$$
$$+ \frac{b}{\cos\theta}\left[\frac{1 + \nu}{2\,\mu}\,\frac{\cos\frac{2}{3}\,\theta}{\cos\frac{\theta}{3}} + \frac{1 - \nu}{2\,\mu}\,\frac{\cos\left(60^\circ - \frac{2}{3}\,\theta\right)}{\cos\left(60^\circ + \frac{\theta}{3}\right)}\right].$$

This expression presents the inconvenience of taking the indeterminate form $\frac{0}{0}$ when the modulus c vanishes and when $\theta = -90^\circ$. This is avoided by putting

$$\nu' = \frac{\sqrt{3}}{64}\,\frac{\nu}{\lambda^2}$$

where we recall that

$$\lambda^2 = \frac{1}{16}\cos\left(60^\circ - \frac{\theta}{3}\right)$$

and transforming the expression into the shape

$$a\,\frac{\sin\left(60^\circ-\frac{\theta}{3}\right)-\nu'}{4\mu\cos^2\frac{\theta}{3}\cos^2\left(60^\circ+\frac{\theta}{3}\right)}+b\,\frac{\frac{1}{2}+\cos\frac{\theta}{3}\cos\left(60^\circ+\frac{\theta}{3}\right)-\nu'\sin\theta}{4\mu\cos^2\frac{\theta}{3}\cos^2\left(60^\circ+\frac{\theta}{3}\right)}.$$

This may be written, if we choose, in the briefer manner

$$a\,\frac{\sin\left(60^\circ-\frac{\theta}{3}\right)-\nu'}{4m^4n^4\mu}+b\,\frac{\frac{1}{2}+m^2n^2-\nu'\sin\theta}{4m^4n^4\mu}.$$

The factors of a and b in this expression are functions of τ, and their common logarithms might be tabulated with τ as the argument.

We will now put

$$\chi(\tau)=\frac{\sin\left(60^\circ-\frac{\theta}{3}\right)-\nu'}{24\sqrt[3]{12\,m^4n^4\mu}}\qquad\qquad \psi(\tau)=\frac{\frac{1}{2}+m^2n^2-\nu'\sin\theta}{24\sqrt[3]{12\,m^4n^4\mu}}$$

as also

$$V=\frac{p}{q}\chi(\tau)+\psi(\tau).$$

Then, if

$$\begin{aligned}
F_1&=\frac{B^2-AC}{3a'^2\cos^2\varphi'.q}\\
F_2&=-\tan\varphi'\cos I.\frac{B\sin\epsilon}{q}\\
F_3&=-\tan\varphi'\sin I.\frac{r}{a}\cos(v+\Pi).\frac{B\sin\epsilon}{q}\\
J_1&=1-\sin^2 I\sin^2(v+\Pi)-\frac{2p}{a'^2\cos^2\varphi'}\\
J_2&=ka\frac{\tan\varphi'}{\cos\varphi'}\frac{r}{a}\sin(v+K)-\tfrac{1}{2}\sin^2 I\sin 2(v+\Pi)\\
J_3&=\sin I\cos I.\frac{r}{a}\sin(v+\Pi)-\alpha\frac{\tan\varphi'}{\cos\varphi'}\sin I\sin\Pi'.\frac{r^2}{a^2}
\end{aligned}$$

where α denotes $\frac{a}{a'}$ we shall have the following equations

$$\begin{aligned}
\frac{a}{r}R_0&=a^2a'^2\cos^2\varphi'.rq^{-\frac{5}{3}}[F_1\chi(\tau)+J_1V]\\
\frac{a}{r}S_0&=a^2a'^2\cos^2\varphi'.rq^{-\frac{5}{3}}[F_2\chi(\tau)+J_2V]\\
\frac{a}{r}W_0&=a^2a'^2\cos^2\varphi'.rq^{-\frac{5}{3}}[F_3\chi(\tau)+J_3V].
\end{aligned}$$

Why we multiply the members of these equations by $\frac{a}{r}$ will presently appear.

A third modification, which seems advantageous, is to apply the process of mechanical quadratures to the quantities $\frac{a}{r} R_0$, $\frac{a}{r} S_0$, and $\frac{a}{r} W_0$ instead of applying it to the variations of the elements. If we multiply the factors of R_0, S_0, and W_0, in the expressions for the variations of the elements, by the factor $\frac{r}{a}$, they become integral functions of sin E and cos E. And thus we have

$$\left[\frac{d\varphi}{dt}\right]_{00} = \frac{m'n}{1+m} M_E\left[\cos\varphi \sin E . \frac{a}{r} R_0 + \left(-\tfrac{3}{2} e + 2\cos E - \frac{e}{2}\cos 2E\right)\frac{a}{r} S_0\right]$$

$$e\left[\frac{d\chi}{dt}\right]_{00} = \frac{m'n}{1+m} M_E\left[-\cos\varphi(\cos E - e)\frac{a}{r} R_0 + \left((2-e^2)\sin E - \frac{e}{2}\sin 2E\right)\frac{a}{r} S_0\right]$$

$$\left[\frac{di}{dt}\right]_{00} = \frac{m'n}{1+m} M_E\left[(-\tan\varphi\cos\omega + \sec\varphi\cos\omega\cos E - \sin\omega\sin E)\frac{a}{r} W_0\right]$$

$$\sin i\left[\frac{d\Omega}{dt}\right]_{00} = \frac{m'n}{1+m} M_E\left[(-\tan\varphi\sin\omega + \sec\varphi\sin\omega\cos E + \cos\omega\sin E)\frac{a}{r} W_0\right]$$

$$\frac{m'n}{1+m} M_E\left[-2\frac{r}{a} R_0\right] = \frac{m'n}{1+m} M_E\left[\left(-(2+e^2) + 4e\cos E - e^2\cos 2E\right)\frac{a}{r} R_0\right].$$

The quantities $\frac{a}{r} R_0$, $\frac{a}{r} S_0$, and $\frac{a}{r} W_0$ by the application of mechanical quadratures, must now be developed in periodic series with the argument E, so that we have

$$\frac{a}{r} R_0 = A_0^{(c)} + A_1^{(c)}\cos E + A_1^{(s)}\sin E + A_2^{(c)}\cos 2E + \ldots\ldots$$

$$\frac{a}{r} S_0 = B_0^{(c)} + B_1^{(c)}\cos E + B_1^{(s)}\sin E + B_2^{(c)}\cos 2E + B_2^{(s)}\sin 2E + \ldots$$

$$\frac{a}{r} W_0 = C_0^{(c)} + C_1^{(c)}\cos E + C_1^{(s)}\sin E + \ldots.$$

where we have written only the terms whose coefficients are needed.

If the circumference, with reference to E, is divided into j parts, and the corresponding values of $\frac{a}{r} R_0$ are $R^{(0)}, R^{(1)}, R^{(2)} \ldots R^{(j-1)}$, then

$$A_0^{(c)} = \frac{1}{j}\left[R^{(0)} + R^{(1)} + R^{(2)} + \ldots + R^{(j-1)}\right]$$

$$\tfrac{1}{2} A_1^{(c)} = \frac{1}{j}\left[R^{(0)} + R^{(1)}\cos\frac{2\pi}{j} + R^{(2)}\cos\frac{4\pi}{j} + \ldots + R^{(j-1)}\cos\frac{2(j-1)\pi}{j}\right]$$

$$\tfrac{1}{2} A_1^{(s)} = \frac{1}{j}\left[R^{(1)}\sin\frac{2\pi}{j} + R^{(2)}\sin\frac{4\pi}{j} + \ldots + R^{(j-1)}\sin\frac{2(j-1)\pi}{j}\right]$$

$$\tfrac{1}{2} A_2^{(c)} = \frac{1}{j}\left[R^{(0)} + R^{(1)}\cos\frac{4\pi}{j} + R^{(2)}\cos\frac{8\pi}{j} + \ldots + R^{(j-1)}\cos\frac{4(j-1)\pi}{j}\right]$$

$$\tfrac{1}{2} A_2^{(s)} = \frac{1}{j}\left[R^{(1)}\sin\frac{4\pi}{j} + R^{(2)}\sin\frac{8\pi}{j} + \ldots + R^{(j-1)}\sin\frac{4(j-1)\pi}{j}\right].$$

Similar equations give the coefficients of $\frac{a}{r} S_0$ and $\frac{a}{r} W_0$.

In fine the following equations result

$$\left[\frac{d\varphi}{dt}\right]_{00} = \frac{m'n}{1+m}\left[\tfrac{1}{2}A_1^{(s)}\cos\varphi - \tfrac{3}{2}eB_0^{(c)} + B_1^{(c)} - \frac{e}{4}B_2^{(c)}\right]$$

$$e\left[\frac{d\chi}{dt}\right]_{00} = \frac{m'n}{1+m}\left[eA_0^{(c)}\cos\varphi - \tfrac{1}{2}A_1^{(c)}\cos\varphi + \left(1-\tfrac{1}{2}e^2\right)B_1^{(s)} - \frac{e}{4}B_2^{(s)}\right]$$

$$\left[\frac{di}{dt}\right]_{00} = \frac{m'n}{1+m}\left[\left(\tfrac{1}{2}C_1^{(c)} - eC_0^{(c)}\right)\sec\varphi\cos\omega - \tfrac{1}{2}C_1^{(s)}\sin\omega\right]$$

$$\sin i\left[\frac{d\Omega}{dt}\right]_{00} = \frac{m'n}{1+m}\left[\left(\tfrac{1}{2}C_1^{(c)} - eC_0^{(c)}\right)\sec\varphi\sin\omega + \tfrac{1}{2}C_1^{(s)}\cos\omega\right]$$

$$\frac{m'n}{1+m}M_E\left[-2\frac{r}{a}R_0\right] = \frac{m'n}{1+m}\left[-(2+e^2)A_0^{(c)} + 2eA_1^{(c)} - \frac{e^2}{2}A_2^{(c)}\right].$$

MEMOIR No. 38

On Certain Possible Abbreviations in the Computation of the Long-Period Inequalities of the Moon's Motion due to the Direct Action of the Planets.

(American Journal of Mathematics, Vol. VI, pp. 115–130, 1883.)

Hansen has characterized the calculation of the coefficients of these inequalities as extremely difficult. However, it seems to me that, if the shortest methods are followed, there is no ground for such an assertion. The work may be divided into two portions, independent of each other. In one the object is to develop, in periodic series, certain functions of the moon's coordinates, which in number do not exceed five. This portion is the same whatever planet may be considered to act, and hence may be done once for all. In the other portion we seek the coefficients of certain terms in the periodic development of certain functions, five also in number, which involve the coordinates of the earth and planet only. And this part of the work is very similar to that in which the perturbations of the earth by the planet in question are the things sought. And as the multiples of the mean motions of these two bodies, which enter into the expression of the argument of the inequalities under consideration, are necessarily quite large, approximative values of the coefficients may be obtained by semi-convergent series similar to the well-known theorem of Stirling. This matter was first elaborated by Cauchy,* but, in the method as left by him, we are directed to compute special values of the successive derivatives of the functions to be developed. Now it unfortunately happens that these functions are enormously complicated by successive differentiation, so that it is almost impossible to write at length their second derivatives. Manifestly then it would be a great saving of labor to substitute for the computation of special values of these derivatives a computation of a certain number of special values of the

* *Mémoire sur les approximations des fonctions de très-grands nombres*; and *Rapport sur un Mémoire de M. Le Verrier, qui a pour objet la détermination d'une grande inégalité du moyen mouvement de la planète Pallas: Comptes Rendus de l'Académie des Sciences de Paris, Tom.* XX, *pp.* 691-726, 767-786, 825-847.

original function, distributed in such a way that the maximum advantage may be obtained. This modification has given rise to an elegant piece of analysis. It will be noticed that, in this method, it is necessary to substitute in the formulæ, from the outset, the numerical values of the elements of the orbits of the earth and planet. There seems to be no objection to this on the practical side, as, for the computation of the inequalities sought, no partial derivatives of R, with respect to these elements, are required.

I.

If the masses of the moon, earth and the planet considered are denoted severally by m, M and m'', and the geocentric rectangular coordinates of the moon by x, y, and z, the similar coordinates of the sun by x', y' and z', and the heliocentric coordinates of the planet by x'', y'' and z'', the perturbative function, for the direct action of the planet on the moon, is

$$R = m''\left[\frac{1}{[(x''+x'-x)^2+(y''+y'-y)^2+(z''+z'-z)^2]^{\frac{1}{2}}} - \frac{(x''+x')x+(y''+y')y+(z''+z')z}{[(x''+x')^2+(y''+y')^2+(z''+z')^2]^{\frac{3}{2}}}\right].$$

But, by a slight substitution in and modification of this expression, we take account of the lunar perturbations of the solar coordinates. Let X, Y and Z denote the coordinates of the sun referred to the centre of gravity of the earth and moon, we shall then have

$$x' = X + \frac{m}{M+m}x, \quad y' = Y + \frac{m}{M+m}y, \quad z' = Z + \frac{m}{M+m}z.$$

And Δ may denote the distance of the planet from the centre of gravity of the earth and moon, so that

$$\Delta^2 = (x''+X)^2 + (y''+Y)^2 + (z''+Z)^2,$$

also r the radius vector of the moon, so that

$$r^2 = x^2 + y^2 + z^2;$$

moreover, for brevity, put

$$P = (x''+X)x + (y''+Y)y + (z''+Z)z.$$

Then R takes the form

$$R = m''\left[\frac{1}{\left[\Delta^2 - 2\frac{M}{M+m}P + \frac{M^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}} - \frac{P + \frac{m}{M+m}r^2}{\left[\Delta^2 + 2\frac{m}{M+m}P + \frac{m^2}{(M+m)^2}r^2\right]^{\frac{3}{2}}}\right].$$

But it is evident that this expression, differentiated with respect to the variables x, y and z, will not furnish differential coefficients identical in value with those the expression gives before the transformation, as x', y' and z' have now been made to involve x, y and z. But a little consideration shows the modification which will remedy this. It is plain we ought to multiply the first term by $\frac{M+m}{M}$, and, multiplying the last term by $-\frac{M+m}{m}$, substitute unity for the numerator and reduce the exponent of the denominator from $\frac{3}{2}$ to $\frac{1}{2}$.

Thus the proper form of R is

$$R = m'' \left[\frac{M+m}{M} \frac{1}{\left[\Delta^2 - 2\frac{M}{M+m} P + \frac{M^2}{(M+m)^2} r^2 \right]^{\frac{1}{2}}} + \frac{M+m}{m} \frac{1}{\left[\Delta^2 + 2\frac{m}{M+m} P + \frac{m^2}{(M+m)^2} r^2 \right]^{\frac{1}{2}}} \right].$$

When this expression is expanded in a series proceeding according to ascending powers of the lunar coordinates, and the terms independent of the latter omitted, we get

$$\begin{aligned} R = m'' \Bigg\{ & \frac{4.3}{2.4} \frac{P^2}{\Delta^5} - \frac{2}{1} \cdot \frac{2.1}{2.4} \frac{r^2}{\Delta^3} \\ & + \frac{M^2 - m^2}{(M+m)^2} \left[\frac{6.5.4}{2.4.6} \frac{P^3}{\Delta^7} - \frac{3}{1} \cdot \frac{4.3.2}{2.4.6} \frac{Pr^2}{\Delta^5} \right] \\ & + \frac{M^3 + m^3}{(M+m)^3} \left[\frac{8.7.6.5}{2.4.6.8} \frac{P^4}{\Delta^9} - \frac{4}{1} \cdot \frac{6.5.4.3}{2.4.6.8} \frac{P^2 r^2}{\Delta^7} + \frac{4.3}{1.2} \cdot \frac{4.3.2.1}{2.4.6.8} \frac{r^4}{\Delta^5} \right] \\ & + \cdots\cdots\cdots \Bigg\}. \end{aligned}$$

The terms of this series follow a quite evident law, and it is easy to write as many as there may be occasion for. But, hitherto, no sensible inequalities have been found arising from the terms beyond the first line. This line furnishes all the inequalities which are not factored by the small ratio $\frac{a}{a'}$, whose value is about $\frac{1}{400}$. And the following two lines of terms can add to the coefficients of these only parts which have the very small factor $\frac{a^2}{a'^2}$. For these reasons we can restrict ourselves to the first line of terms, and write very simply

$$R = m'' \left[\frac{3}{2} \frac{P^2}{\Delta^5} - \frac{1}{2} \frac{r^2}{\Delta^3} \right].$$

Restoring the equivalent of P,

$$R = m'' \left\{ \left[\frac{3}{2} \frac{(x'' + X)^2}{\Delta^5} - \frac{1}{2} \frac{1}{\Delta^3} \right] x^2 + \left[\frac{3}{2} \frac{(y'' + Y)^2}{\Delta^5} - \frac{1}{2} \frac{1}{\Delta^3} \right] y^2 \right.$$
$$+ \left[\frac{3}{2} \frac{(z'' + Z)^2}{\Delta^5} - \frac{1}{2} \frac{1}{\Delta^3} \right] z^2 + 3 \frac{(x'' + X)(y'' + Y)}{\Delta^5} xy$$
$$\left. + 3 \frac{(x'' + X)(z'' + Z)}{\Delta^5} xz + 3 \frac{(y'' + Y)(z'' + Z)}{\Delta^5} yz \right\}.$$

This expression has the advantage of exhibiting the value of R as a sum of terms of which each is the product of two factors, one of which depends solely on the coordinates of the moon and the other is independent of them.

If we denote the factors of x^2, y^2 and z^2 in R severally by A, B and C, we shall have the relation $A + B + C = 0$.
Hence it is plane that the number of terms can be reduced from six to five. As we shall take the ecliptic for the plane of xy, we will have $Z = 0$. We can then write

$$R = m'' \left\{ \frac{1}{4} \left[\frac{1}{\Delta^3} - 3 \frac{z''^2}{\Delta^5} \right] (r^2 - 3z^2) + \frac{3}{4} \frac{(y'' + Y)^2 - (x'' + X)^2}{\Delta^5} (y^2 - x^2) \right.$$
$$\left. + 3 \frac{(x'' + X)(y'' + Y)}{\Delta^5} xy + 3 \frac{(x'' + X)z''}{\Delta^5} xz + 3 \frac{(y'' + Y)z''}{\Delta^5} yz \right\}.$$

II.

We will now express the five factors of the terms of R, viz. $r^2 - 3z^2$, $x^2 - y^2$, xy, xz and yz, as functions of t, the time, when elliptic values are attributed to the coordinates, leaving, however, the longitudes of the perigee and node indeterminate, so that the latter may have their motions proportional to t.

Using Delaunay's notation, and, in addition, putting v for the true anomaly, we have

$$x = r \cos (v + g) \cos h - (1 - 2\gamma^2) r \sin (v + g) \sin h,$$
$$y = r \cos (v + g) \sin h + (1 - 2\gamma^2) r \sin (v + g) \cos h,$$
$$z = 2\gamma \sqrt{1 - \gamma^2}\, r \sin (v + g);$$

or, in a slightly different form,

$$x = (1 - \gamma^2) r \cos (v + g + h) + \gamma^2 r \cos (v + g - h),$$
$$y = (1 - \gamma^2) r \sin (v + g + h) - \gamma^2 r \sin (v + g - h),$$
$$z = 2\gamma \sqrt{1 - \gamma^2}\, r \sin (v + g).$$

From these equations we derive

$$
\begin{aligned}
z^2 &= 2\gamma^2(1-\gamma^2)\,r^2[1-\cos 2(v+g)], \\
r^2 - 3z^2 &= [1-6\gamma^2+6\gamma^4]\,r^2 + 6\gamma^2(1-\gamma^2)\,r^2\cos 2(v+g), \\
x^2 - y^2 &= (1-\gamma^2)^2 r^2\cos 2(v+g+h) + \gamma^4 r^2\cos 2(v+g-h) + 2\gamma^2(1-\gamma^2)\,r^2\cos 2h, \\
2xy &= (1-\gamma^2)^2 r^2\sin 2(v+g+h) - \gamma^4 r^2\sin 2(v+g-h) + 2\gamma^2(1-\gamma^2)\,r^2\sin 2h, \\
xz &= \gamma(1-\gamma^2)^{\frac{3}{2}} r^2\sin(2v+2g+h) + \gamma^3(1-\gamma^2)^{\frac{1}{2}} r^2\sin(2v+2g-h) \\
&\qquad - \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} r^2\sin h, \\
yz &= -\gamma(1-\gamma^2)^{\frac{3}{2}} r^2\cos(2v+2g+h) + \gamma^3(1-\gamma^2)^{\frac{1}{2}} r^2\cos(2v+2g-h) \\
&\qquad + \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} r^2\cos h.
\end{aligned}
$$

It is then plain that the development of these five factors depends on that of the quantities r^2, $r^2\cos 2v$ and $r^2\sin 2v$. Denoting the eccentric anomaly by u, we have

$$
\begin{aligned}
\frac{r^2}{a^2} &= (1-e\cos u)^2, \\
\frac{r^2}{a^2}\cos 2v &= \frac{3}{2}e^2 - 2e\cos u + \left(1-\frac{1}{2}e^2\right)\cos 2u, \\
\frac{r^2}{a^2}\sin 2v &= \sqrt{1-e^2}\,(\sin 2u - 2e\sin u).
\end{aligned}
$$

The constant terms of these functions, in their development in periodic series involving multiples of the mean anomaly, are the same as the constant terms of the right members of the last equations after they have been multiplied by $1-e\cos u$. That is, these terms are severally $1+\frac{3}{2}e^2$, $\frac{5}{2}e^2$ and 0. To obtain the remaining coefficients, we put $s=\varepsilon^{u\sqrt{-1}}$, and $z=\varepsilon^{l\sqrt{-1}}$, and recall the theorem that the coefficient of z^i, in the development of any function S according to powers of z, is the same as that of s^i in the development of

$$\frac{s}{i}\frac{dS}{ds}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)},$$

according to powers of s. Moreover, adopting Hansen's notation for the Besselian function, we put $\varepsilon^{\lambda\left(s-\frac{1}{s}\right)} = \Sigma_i . J_\lambda^{(i)} s^i$,
so that, for positive values of i, we have

$$J_\lambda^{(i)} = \frac{\lambda^i}{1.2\ldots i}\left[1-\frac{\lambda^2}{1.(i+1)}+\frac{\lambda^4}{1.2(i+1)(i+2)}-\ldots\right],$$

and, for negative values, $\qquad J_\lambda^{(-i)} = J_{-\lambda}^{(i)}.$

These functions satisfy the following equation,

$$iJ_\lambda^{(i)} = \lambda\,(J_\lambda^{(i-1)} + J_\lambda^{(i+1)}).$$

Whence

$$J_\lambda^{(i-1)} = \frac{i}{\lambda}\,J_\lambda^{(i)} - J_\lambda^{(i+1)},$$

$$J_\lambda^{(i+1)} = \frac{i}{\lambda}\,J_\lambda^{(i)} - J_\lambda^{(i-1)},$$

and, by writing $i - 1$ for i in the first of these and $i + 1$ for i in the second,

$$J_\lambda^{(i-2)} = \frac{i-1}{\lambda}\,J_\lambda^{(i-1)} - J_\lambda^{(i)},$$

$$J_\lambda^{(i+2)} = \frac{i+1}{\lambda}\,J_\lambda^{(i+1)} - J_\lambda^{(i)}.$$

Consequently

$$J_\lambda^{(i-2)} - J_\lambda^{(i+2)} = \frac{1}{\lambda}\,[(i-1)\,J_\lambda^{(i-1)} - (i+1)\,J_\lambda^{(i+1)}].$$

The coefficient of z^i in the expansion of $\frac{r^2}{a^2}$ being equal to that of s^i in

$$-\frac{e}{i}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]\left(s - \frac{1}{s}\right)\varepsilon^{\frac{ie}{2}\left(s - \frac{1}{s}\right)},$$

is

$$-\frac{e}{i}\left[J_{\frac{ie}{2}}^{(i-1)} - J_{\frac{ie}{2}}^{(i+1)} - \frac{e}{2}\left(J_{\frac{ie}{2}}^{(i-2)} - J_{\frac{ie}{2}}^{(i+2)}\right)\right],$$

which, by means of the relations between the J functions just given, reduces to

$$-\frac{2}{i^2}\,J_{\frac{ie}{2}}^{(i)}.$$

Hence we have

$$\frac{r^2}{a^2} = 1 + \frac{3}{2}\,e^2 - \sum_{i=1}^{i=\infty} . \frac{4}{i^2}\,J_{\frac{ie}{2}}^{(i)} \cos il.$$

This result may also be obtained from the equation

$$\frac{d^2\frac{r^2}{a^2}}{dl^2} = 2\,\frac{a}{r} - 2.$$

In like manner we get

$$\frac{r^2}{a^2}\cos 2v = \frac{5}{2}\,e^2 + \sum_{i=1}^{i=\infty} . \frac{2}{i}\left[\left(1 - \frac{1}{2}\,e^2\right)\left(J_{\frac{ie}{2}}^{(i-2)} - J_{\frac{ie}{2}}^{(i+2)}\right) - e\left(J_{\frac{ie}{2}}^{(i-1)} - J_{\frac{ie}{2}}^{(i+1)}\right)\right]\cos il,$$

$$\frac{r^2}{a^2}\sin 2v = \sqrt{1-e^2}\sum_{i=1}^{i=\infty} . \frac{2}{i}\left[J_{\frac{ie}{2}}^{(i-2)} + J_{\frac{ie}{2}}^{(i+2)} - e\left(J_{\frac{ie}{2}}^{(i-1)} + J_{\frac{ie}{2}}^{(i+1)}\right)\right]\sin il.$$

Consequently, if we put

$$H^{(i)} = \frac{2}{i}\Big[\left(\cos^2\frac{\varphi}{2} - \frac{1}{4}\,e^2\right)J_{\frac{ie}{2}}^{(i-2)} - e\cos^2\frac{\varphi}{2}\,.\,J_{\frac{ie}{2}}^{(i-1)}$$
$$+ e\sin^2\frac{\varphi}{2}\,.\,J_{\frac{ie}{2}}^{(i+1)} - \left(\sin^2\frac{\varphi}{2} - \frac{1}{4}\,e^2\right)J_{\frac{ie}{2}}^{(i+2)}\Big],$$

where $\sin \phi = e$, and we agree that

$$H^{(0)} = \frac{5}{2} e^2,$$

we shall have, α denoting any arbitrary angle,

$$r^2 \cos(\alpha + 2v) = a^2 \sum_{i=-\infty}^{i=+\infty} . H^{(i)} \cos(\alpha + il),$$

$$r^2 \sin(\alpha + 2v) = a^2 \sum_{i=-\infty}^{i=+\infty} . H^{(i)} \sin(\alpha + il).$$

We can now write the expansions of the five factors of the terms of R which depend solely on the moon's coordinates:

$$\begin{aligned}
\frac{r^2 - 3z^2}{4a^2} &= -\frac{1}{2}(1 - 6\gamma^2 + 6\gamma^4)\Sigma . \frac{1}{i^2} J^{(i)}_{\frac{ie}{2}} \cos il \\
&\quad + \frac{3}{2}\gamma^2(1-\gamma^2)\Sigma . H^{(i)} \cos(2g + il), \\
\frac{3}{4}\frac{x^2 - y^2}{a^2} &= \frac{3}{4}(1-\gamma^2)^2\Sigma . H^{(i)} \cos(2h + 2g + il) \\
&\quad - 3\gamma^2(1-\gamma^2)\Sigma . \frac{1}{i^2} J^{(i)}_{\frac{ie}{2}} \cos(2h + il) \\
&\quad + \frac{3}{4}\gamma^4\Sigma . H^{(i)} \cos(-2h + 2g + il), \\
\frac{3}{2}\frac{xy}{a^2} &= \frac{3}{4}(1-\gamma^2)^2\Sigma . H^{(i)} \sin(2h + 2g + il) \\
&\quad - 3\gamma^2(1-\gamma^2)\Sigma . \frac{1}{i^2} J^{(i)}_{\frac{ie}{2}} \sin(2h + il) \\
&\quad - \frac{3}{4}\gamma^4\Sigma . H^{(i)} \sin(-2h + 2g + il), \\
\frac{3}{2}\frac{xz}{a^2} &= \frac{3}{2}\gamma(1-\gamma^2)^{\frac{1}{2}}\Sigma . H^{(i)} \sin(h + 2g + il) \\
&\quad + 3\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\Sigma . \frac{1}{i^2} J^{(i)}_{\frac{ie}{2}} \sin(h + il) \\
&\quad + \frac{3}{2}\gamma^3(1-\gamma^2)^{\frac{1}{2}}\Sigma . H^{(i)} \sin(-h + 2g + il), \\
\frac{3}{2}\frac{yz}{a^2} &= -\frac{3}{2}\gamma(1-\gamma^2)^{\frac{1}{2}}\Sigma . H^{(i)} \cos(h + 2g + il) \\
&\quad - 3\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\Sigma . \frac{1}{i} J^{(i)}_{\frac{ie}{2}} \cos(h + il) \\
&\quad + \frac{3}{2}\gamma^3(1-\gamma^2)^{\frac{1}{2}}\Sigma . H^{(i)} \cos(-h + 2g + il).
\end{aligned}$$

The summation must be extended to all integral values positive and negative, zero included, for i. When $i = 0$ we must suppose that $\frac{1}{i^2} J^{(i)}_{\frac{ie}{2}}$ takes the value $-\frac{1}{2}\left(1 + \frac{3}{2}e^2\right)$.

It will be perceived that the three first terms of R furnish inequalities whose arguments do not involve the longitude of the moon's node or involve it in an even multiple. The two remaining terms furnish inequalities having an odd multiple of this longitude in their arguments. And it is evident that these statements remain true even when the solar perturbations of the lunar coordinates are taken into consideration. Hence, in deriving any particular inequality, we never have to consider more than three out of the five terms of R. When we propose to neglect the solar perturbations, it can be seen at a glance what terms of the expressions above ought to be retained. Thus, in the case of Hansen's inequality of 273 years, the argument involving only l without either h or g, it is plain tha the first term of $\frac{r^2 - 3z^2}{4a^3}$ can alone furnish it; and consequently, we may put, very simply,

$$R = - m''a^2 (1 - 6\gamma^2 + 6\gamma^4) J_{\frac{e}{2}}^{(1)} \left[\frac{1}{\Delta^3} - 3\frac{z''^2}{\Delta^5}\right] \cos l.$$

And the whole difficulty is reduced to finding, in the development of

$$\frac{1}{\Delta^3} - 3\frac{z''^2}{\Delta^5},$$

the terms $$A^{(c)} \cos (18l'' - 16l') + A^{(s)} \sin (18l'' - 16l').$$

III.

We pass now to the consideration of the development, in periodic series, of the factors of the terms of R which depend on the coordinates of the earth and planet. Let it be required to discover the coefficient $C_{i,i'}$ of $z^i z'^{i'}$ in the development of any periodic function of the eccentric anomalies u and u' of two planets, in the case where i is quite large. We shall suppose that the function has $\frac{1}{\Delta^{2n}}$ for a factor. It is known that

$$\frac{1}{\Delta^{2n}} = N^{2n} [1 - 2\mathfrak{a} \cos (u - Q) + \mathfrak{a}^2]^{-n} [1 - 2\mathfrak{b} \cos (u + Q) + \mathfrak{b}^2]^{-n},$$

where N, $\mathfrak{a}$, $\mathfrak{b}$ and Q are functions of u' or l' only, and $\mathfrak{a}$ and $\mathfrak{b}$ are always less than unity. Substituting the imaginary exponential $s = \varepsilon^{u\sqrt{-1}}$, and, to abbreviate, putting $k = \mathfrak{a}^{-1} \varepsilon^{Q\sqrt{-1}}$, $k_1 = \mathfrak{b}^{-1} \varepsilon^{-Q\sqrt{-1}}$, this equation becomes

$$\frac{1}{\Delta^{2n}} = N^{2n} \left(1 - \frac{s}{k}\right)^{-n} \left(1 - \frac{\mathfrak{a}^2 k}{s}\right)^{-n} \left(1 - \frac{s}{k_1}\right)^{-n} \left(1 - \frac{\mathfrak{b}^2 k_1}{s}\right)^{-n}.$$

Rendering evident the factor $\left(1-\frac{s}{k}\right)^{-n}$, we can then suppose that the function to be developed is

$$\left(1-\frac{s}{k}\right)^{-n} F(s).$$

The coefficient of z^i in the development of this is equivalent to

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} s^{-i}\varepsilon^{\frac{\mu}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]\left(1-\frac{s}{k}\right)^{-n} F(s)\,du.$$

Let us put $$f(s) = \varepsilon^{\frac{\mu}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right] F(s);$$

then $$C_i = \frac{1}{2\pi}\int_0^{2\pi} s^{-i}\left(1-\frac{s}{k}\right)^{-n} f(s)\,du.$$

Since the absolute term of a series of integral powers of a variable is not changed by substituting for the latter a constant multiple of it, in the expression for C_i we can write ks for s. Thus

$$C_i = \frac{k^{-i}}{2\pi}\int_0^{2\pi} s^{-i}(1-s)^{-n} f(ks)\,du.$$

The difficulty here that the factor $(1-s)^{-n}$ becomes infinite at the limits of the definite integral, is only apparent. For the multiple of s instead of ks may be ps, in which the modulus of p is less than that of k by a very small quantity. In this case we get a tangible result, which is seen to have, as its limit, when p is made to approach k indefinitely, the value which will be presently given.

We now assume that it is possible to expand $f(ks)$ in an infinite series proceeding according to positive integral powers of u.* Let

$$f(ks) = c_0 + c_1 u + c_2 u^2 + \ldots = \Sigma . c_j u^j.$$

Then $$C_i = \frac{k^{-i}}{2\pi}\Sigma . \int_0^{2\pi}\varepsilon^{-iu\sqrt{-1}}\left(1-\varepsilon^{u\sqrt{-1}}\right)^{-n} c_j u^j\,du.$$

The definite integral $$\frac{1}{2\pi}\int_0^{2\pi}\varepsilon^{-iu\sqrt{-1}}\left(1-\varepsilon^{u\sqrt{-1}}\right)^{-n} du$$

is a function of n and i: with Cauchy we will denote it by $[n]_i$. Then by taking the derivative of the quantity, under the integral sign, j times with respect to i, we get $\frac{1}{2\pi}\int_0^{2\pi}\varepsilon^{-iu\sqrt{-1}}\left(1-\varepsilon^{u\sqrt{-1}}\right)^{-n} u^j\,du = (\sqrt{-1})^j D_i^j . [n]_i$.

* This is the assumption which leads to the semi-convergent series representing the value of C_i. Its allowableness is shown by the fact of the relative smallness of the definite integral which ought to be added to complete the truncated series, when i is tolerably large and the number of terms taken into account is not too great. As Cauchy has treated this point at length, in his memoir first mentioned above, I have thought it unnecessary to say more about it here.

Whence we have the symbolic expression for C_i,

$$C_i = k^{-i} f(k\varepsilon^{-D_i}) \,.\, [n]_i \,.$$

But we have

$$\varepsilon^{D_i} = 1 + \Delta \,,\quad \varepsilon^{-D_i} = \frac{1}{1+\Delta}$$

Δ here denoting the characteristic of finite differences with respect to the variable i, and not the distance between the two planets. Let

$$\nabla = \frac{\Delta}{1+\Delta}, \text{ then } \varepsilon^{-D_i} = 1 - \nabla \,.$$

Making these substitutions, we have

$$C_i = k^{-i} f(k - k\nabla) \,.\, [n]_i \,.$$

By successive integrations by parts, making the integration always bear on the first factor, we find the value of the definite integral,

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}} (1 - \varepsilon^{u\sqrt{-1}})^{-n} du = [n]_i = \frac{n(n+1)\ldots(n+i-1)}{1.2\ldots i} \,.$$

When the function $f(k - k\nabla)$ is developed in ascending powers of ∇, the general term of C_i will be proportional to

$$\nabla^j \,.\, [n]_i = \frac{\Delta^j}{(1+\Delta)^j} \,.\, [n]_i = \Delta^j \,.\, [n]_{i-j} = [n-j]_i \,.$$

And, developing the last expression for C_i, and employing accents, attached to f, to denote differentiation of the form of f, we have

$$C_i = k^{-i}\left\{ f(k)[n]_i - kf'(k)[n-1]_i + \frac{1}{1.2}k^2 f''(k)[n-2] \right.$$
$$\left. - \frac{1}{1.2.3}\, k^3 f'''(k)[n-3]_i + \ldots \right\} .$$

This may also be written

$$C_i = k^{-i}[n]_i \left\{ f(k) - f'(k) \,.\, k\frac{n-1}{i+n-1} \right.$$
$$+ \frac{1}{1.2} f''(k) \,.\, k^2 \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}$$
$$- \frac{1}{1.2.3} f'''(k) \,.\, k^3 \frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)}$$
$$\left. + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\} .$$

We may employ the Γ function to express $[n]_i$, and then

$$[n]_i = \frac{\Gamma(i+n)}{\Gamma(n)\Gamma(i+1)} \,.$$

In practice, n will have some one of the following series of values,

$$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},\ \text{etc.};$$

and it is well known that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},\ \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi},\ \Gamma\left(\frac{5}{2}\right)=\frac{3}{4}\sqrt{\pi},\ \text{etc.}$$

When i is a tolerably large integer, we may use the semi-convergent series

$$\log\Gamma(i+n)=\frac{1}{2}\log(2\pi)+\left(i+n-\frac{1}{2}\right)\log(i+n-1)$$
$$+M\left\{-(i+n-1)+\frac{B_1}{1.2}\frac{1}{i+n-1}-\frac{B_3}{3.4}\frac{1}{(i+n-1)^3}+\frac{B_5}{5.6}\frac{1}{(i+n-1)^5}-\ldots\right\},$$
$$\log\Gamma(i+1)=\frac{1}{2}\log(2\pi)+\left(i+\frac{1}{2}\right)\log i$$
$$+M\left\{-i+\frac{B_1}{1.2}\frac{1}{i}-\frac{B_3}{3.4}\frac{1}{i^3}+\frac{B_5}{5.6}\frac{1}{i^5}-\ldots\right\},$$

where M is the modulus of common logarithms, and B_1, B_3, etc., are the numbers of Bernoulli. Thence is derived

$$\begin{aligned}\log\frac{\Gamma(i+n)}{\Gamma(i+1)}&=\left(i+\frac{1}{2}\right)\log\frac{i+n-1}{i}+(n-1)\log(i+n-1)\\&-M\left\{n-1+\frac{B_1}{1.2}\left[\frac{1}{i}-\frac{1}{i+n-1}\right]-\frac{B_3}{3.4}\left[\frac{1}{i^3}-\frac{1}{(i+n-1)^3}\right]\right.\\&\left.+\frac{B_5}{5.6}\left[\frac{1}{i^5}-\frac{1}{(i+n-1)^5}\right]-\ldots\right\}\\&=\left(i+\frac{1}{2}\right)\log\frac{i+n-1}{i}+(n-1)\log(i+n-1)\\&-M(n-1)\left\{1+\frac{1}{12}\frac{1}{i(i+n-1)}-\frac{1}{360}\frac{i^2+i(i+n-1)+(i+n-1)^2}{i^3(i+n-1)^3}\right.\\&+\frac{1}{1260}\frac{i^4+i^3(i+n-1)+i^2(i+n-1)^2+i(i+n-1)^3+(i+n-1)^4}{i^5(i+n-1)^5}\\&\left.-\ldots\ldots\ldots\ldots\ldots\ldots\right\}.\end{aligned}$$

The first term of the last expression for C^i affords a first approximation to its value, correct, so to speak, to quantities of the order of $\frac{1}{i}$. Then

$$C_i=k^{-i}[n]_i f(k).$$

In like manner, the two terms at the beginning afford an approximation correct to quantities of the order of $\frac{1}{i^2}$. Here we can effect a remarkable

reduction; for on comparing the two terms in question with the two first terms of Taylor's theorem, we see that, to the same degree of approximation, we may write

$$C_i = k^{-i}[n]_i f\left(\frac{i}{i+n-1}k\right).$$

No more labor is involved in employing this expression than in the preceding.

IV.

In this condition Cauchy leaves the subject, but we may go a step farther. In the cases which come up in practice $f(k)$ is always such a function that successive differentiation immensely complicates it; so that it is scarcely possible to go beyond $f''(k)$. Hence a great deal of labor is saved, if, instead of attempting to calculate $f'(k), f''(k)$, etc., we substitute the calculation of $f(k)$ for several values of the argument k. It is easy to perceive that, in general, all the derivatives $f'(k), f''(k)$, etc., may be eliminated from the expression for C_i. For, cutting the series off at the term which contains $f^{(2p)}(k)$ as a factor, we may suppose that, to the same degree of approximation,

$$C_i = k^{-i}[n]_i\{x_0 f(k-ky_0) + x_1 f(k-ky_1) + \ldots + x_p f(k-ky_p)\},$$

where $x_0, x_1, \ldots x_p$ and $y_0, y_1, \ldots y_p$ are unknowns to be suitably determined.

By developing this expression for C_i in powers of k and comparing it with the previous expression, we get the following system of simultaneous equations for determining the unknowns $x_0, x_1, \ldots x_p, y_0, y_1, \ldots y_p$:

$$\begin{aligned}
x_0 + x_1 + x_2 + \ldots + x_p &= 1,\\
x_0y_0 + x_1y_1 + x_2y_2 + \ldots + x_py_p &= \frac{(n-1)}{i+n-1},\\
x_0y_0^2 + x_1y_1^2 + x_2y_2^2 + \ldots + x_py_p^2 &= \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)},\\
&\ldots\ldots\ldots\ldots\\
x_0y_0^{2p+1} + x_1y_1^{2p+1} + x_2y_2^{2p+1} + \ldots + x_py_p^{2p+1} &= \frac{(n-1)\ldots(n-2p-1)}{(i+n-1)\ldots(i+n-2p-1)}.
\end{aligned}$$

For the sake of brevity we will denote the right-hand members of these equations as $a_0, a_1, a_2 \ldots a_{2p+1}$. The solution of these equations is very elegant. According to the theorem of Bezout, the degree of the final equation, obtained by elimination, would be $= (2p+2)!$ But as we shall see, the solution depends on that of an equation of the $(p+1)^{\text{th}}$ degree, whose roots are the values of the several unknowns $y_0, y_1 \ldots y_p$; and there is practically but one solution.

Let us suppose that the values of the y's, in any particular solution, are the roots of the equation

$$y^{p+1} + s_1 y^p + s_2 y^{p-1} + \ldots + s_{p+1} = 0,$$

so that

$$s_1 = -(y_0 + y_1 + y_2 + \ldots + y_p),$$
$$s_2 = y_0 y_1 + y_0 y_2 + y_1 y_2 + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_{p+1} = (-1)^{p+1} y_0 y_1 y_2 \ldots y_p .$$

Hence, y_q denoting any one of the y's, we must have

$$y_q^{p+1} + s_1 y^p + s_2 y_q^{p-1} + \ldots + s_{\;+1} = 0.$$

Now, in the group of equations to be solved, multiply the equation, whose second member is a_{p+1}, by 1, the one, whose second member is a_p, by s_1, and so on until the first equation is multiplied by s_{p+1}. Then, by adding all the equations thus obtained, the first member of the resulting equation vanishes, and we have $a_{p+1} + a_p s_1 + a_{p-1} s_2 + \ldots + a_0 s_{p+1} = 0$.
By cutting off the first equation and adding to the group the equation whose second member is a_{p+2}, and writing x_0 for $x_0 y_0$, x_1 for $x_1 y_1$, and so on, we obtain a group which differs from the former only in the second members. Hence we have, from this group, the equation

$$a_{p+2} + a_{p+1} s_1 + a_p s_2 + \ldots + a_1 s_{p+1} = 0.$$

And, in a similar manner,

$$a_{p+3} + a_{p+2} s_1 + a_{p+1} s_2 + \ldots + a_2 s_{p+1} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{2p+1} + a_{2p} s_1 + a_{2p-1} s_2 + \ldots + a_p s_{p+1} = 0.$$

These $p + 1$ linear equations suffice to determine the values of s_1, $s_2 \ldots s_{p+1}$, the coefficients of the equation of the $(p+1)^{\text{th}}$ degree, which has, as its roots, the values of the unknowns y. These values, being obtained and substituted in the first $(p+1)$ equations of the original group, we have a group of $(p+1)$ linear equations for determining the $(p+1)$ unknowns $x_0, x_1 \ldots x_p$. It is plain that all possible solutions of the group of equations are obtained by permuting between themselves the roots of the equation which gives the values of the y's; and, as thus, to each root, corresponds its special value of the x's, and the order in which the several terms of C_i stand, is of no import, it is clear that, practically at least, but one solution exists.

In practice, p never need exceed 2. For $p = 0$, the solution has already been given. For $p = 1$, we have

$$\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)} + \frac{n-1}{i+n-1}\,s_1 + s_2 = 0\,,$$

$$\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)} + \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}\,s_1 + \frac{n-1}{i+n-1}\,s_2 = 0\,.$$

The solution of these gives

$$s_1 = -2\,\frac{n-2}{i+n-3}\,, \quad s_2 = \frac{(n-1)(n-2)}{(i+n-2)(i+n-3)}\,.$$

Thus the equation which contains the values of the y's is

$$y^2 - 2\,\frac{n-2}{i+n-3}\,y + \frac{(n-1)\,(n-2)}{(i+n-2)\,(i+n-3)} = 0\,.$$

Whence the two values of y are

$$y = \frac{n-2 \pm \sqrt{\dfrac{(2-n)(i-1)}{i+n-2}}}{i+n-3}\,;$$

and the corresponding values of x are

$$x = \tfrac{1}{2}\left[1 \pm \frac{i-n+1}{i+n-1}\sqrt{\frac{i+n-2}{(2-n)(i-1)}}\,\right].$$

In many cases these values will be imaginary, which, however, does not hinder their use, as k is imaginary.

For $p = 2$, we have

$$\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)} + \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}\,s_1 + \frac{n-1}{i+n-1}\,s_2 + s_3 = 0\,,$$

$$\frac{(n-2)(n-3)(n-4)}{(i+n-2)(i+n-3)(i+n-4)} + \frac{(n-2)(n-3)}{(i+n-2)(i+n-3)}\,s_1 + \frac{n-2}{i+n-2}\,s_2 + s_3 = 0\,,$$

$$\frac{(n-3)(n-4)(n-5)}{(i+n-3)(i+n-4)(i+n-5)} + \frac{(n-3)(n-4)}{(i+n-3)(i+n-4)}\,s_1 + \frac{n-3}{i+n-3}\,s_2 + s_3 = 0\,.$$

The solution of these equations gives

$$s_1 = -3\,\frac{n-3}{i+n-5}\,, \; s_2 = 3\,\frac{(n-2)(n-3)}{(i+n-4)(i+n-5)}\,, \; s_3 = -\,\frac{(n-1)(n-2)(n-3)}{(i+n-3)(i+n-4)(i+n-5)}\,.$$

The equation, which has, for its roots, the values of the y's, is

$$y^3 - 3\,\frac{n-3}{i+n-5}\,y^2 + 3\,\frac{(n-2)(n-3)}{(i+n-4)(i+n-5)}\,y - \frac{(n-1)(n-2)(n-3)}{(i+n-3)(i+n-4)(i+n-5)} = 0\,.$$

By comparing this with the equation for the case where $p=1$, we readily see what the equation would be for higher values of p.

As an example, suppose it were required to find the coefficient of z^{18} in the expansion of
$$[1-2\mathfrak{a}\cos(u-Q)+\mathfrak{a}^2]^{-\frac{3}{2}}.$$

Here the form of $f(s)$ is

$$f(s)=\left(1-\frac{\mathfrak{a}^2k}{s}\right)^{-\frac{3}{2}}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]\epsilon^{iu\left(s-\frac{1}{s}\right)}.$$

In the first place let two terms in the final expression for C_i be regarded as sufficient, that is, put $p=1$. Then $i=18$, $n=\frac{3}{2}$, and the two values of y are

$$y=\frac{-1\pm 2\sqrt{\frac{17}{35}}}{33};$$

and the corresponding value of x is

$$x=\frac{1}{2}\left(1\pm\frac{35}{37}\sqrt{\frac{35}{17}}\right).$$

Thus the expression for C_i is

$$C_{18}=k^{-18}\left[\frac{3}{2}\right]_{18}\{1.17865f(0.9880647k)-0.17865f(1.0725413k)\}.$$

The error of this is of the order of $\frac{1}{i^4}$, while, in case $p=0$, which gives the formula

$$C_{18}=k^{-18}\left[\frac{3}{2}\right]_{18}f\left(\frac{36}{37}k\right),$$

which Cauchy employed, the error is of the order of $\frac{1}{i^3}$.

In case we make $p=2$, and thus have three terms in the formula for C_i, the roots of the cubic

$$y^3+\frac{9}{29}y^2+\frac{9}{31.29}y-\frac{3}{33.31.29}=0$$

must be found. They are

$$y_0=+0.00804343,\quad y_1=-0.04617994,\quad y_2=-0.27220828.$$

The linear equations for determining the x's are

$$\begin{aligned}
x_0 + \qquad x_1 + \qquad x_2 &= 1,\\
0.0804343x_0-0.4617994x_1-2.722083x_2&=0.2702703,\\
0.0064697x_0+0.2132586x_1+7.409736x_2&=-0.0772201.
\end{aligned}$$

The solution of which gives

$$x_0 = +1.3426685, \quad x_1 = -0.3408857, \quad x_2 = -0.0017828.$$

Thus, in this case, we should have

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18}\{1.3426685 f(0.9919566k) - 0.3408857 f(1.04617994k) - 0.0017828 f(1.2722083k)\}.$$

The error of this formula is only of the order of $\frac{1}{i^6}$.

In further illustration of this method, let us find the value $\mathfrak{b}_{\frac{1}{2}}^{(18)}$ the coefficient of cos 18θ in the periodic development of

$$(1 - 2\alpha \cos\theta + \alpha^2)^{-\frac{1}{2}},$$

where $\alpha = 0.723332$ the ratio of the mean distances of Venus and the earth from the sun. Here the form of $f(s)$ is simply

$$f(s) = \left(1 - \frac{\alpha}{s}\right)^{-\frac{1}{2}}.$$

Let us take the formula where $p = 1$. We have

$$\mathfrak{b}_{\frac{1}{2}}^{(18)} = 2C_{18} = 2\left[\frac{3}{2}\right]_{18}\alpha^{18}\left\{1.17865\left(1 - \frac{\alpha^2}{0.9880647}\right)^{-\frac{1}{2}} - 0.17865\left(1 - \frac{\alpha^2}{1.0725413}\right)^{-\frac{1}{2}}\right\}.$$

The value of $\left[\frac{3}{2}\right]_{18}$ will be found in the table at the end of this memoir. And on the substitution of the numerical values, we get $\mathfrak{b}_{\frac{1}{2}}^{(18)} = 0.090880$. Delaunay, in his memoir,* has 0.090876.

In the case where the function to be developed contains the anomalies of two planets, after the value of C_i has been obtained corresponding to j points evenly distributed on the circumference with reference to the variable ϵ' or the variable u', the value of $C_{i,i'}$ results by employing the method of mechanical quadratures: the formula in the first case being

$$C_{i,i'} = \frac{1}{j}\Sigma.\, C_i z'^{-i'},$$

and, in the second,

$$C_{i,i'} = \frac{1}{j}\Sigma.\, C_i \frac{r'}{a'} s'^{-i'}.$$

In the annexed table are given the common logarithms of the function $[n]_i$, for n as far as $n = \frac{9}{2}$, and for i, as far as $i = 30$. As they have been computed with the ten-figure logarithms of Vega's *Thesaurus Logarithmorum*, it is to be presumed that they are correct, in nearly every case, to half a unit in the last place.

* *Connaissance des Temps*, 1862.

TABLE OF THE VALUES OF LOG $[n]_i$.

i.	$n = \frac{1}{2}$.	$n = \frac{3}{2}$.	$n = \frac{5}{2}$.	$n = \frac{7}{2}$.	$n = \frac{9}{2}$.
1	9.6989700	0.1760913	0.3979400	0.5440680	0.6532125
2	9.5740313	0.2730013	0.6409781	0.8962506	1.0925452
3	9.4948500	0.3399481	0.8170693	1.1594920	1.4283373
4	9.4368581	0.3911006	0.9553720	1.3703454	1.7013386
5	9.3911006	0.4324933	1.0693154	1.5464366	1.9317875
6	9.3533120	0.4672554	1.1662254	1.6977043	2.1313599
7	9.3211273	0.4972186	1.2505463	1.8303299	2.3074511
8	9.2930986	0.5235475	1.3251799	1.9484292	2.4650590
9	9.2682750	0.5470286	1.3921267	2.0548845	2.6077265
10	9.2459986	0.5682179	1.4528245	2.1517945	2.7380602
11	9.2257953	0.5875231	1.5083418	2.2407356	2.8580356
12	9.2073118	0.6052519	1.5594944	2.3229224	2.9691860
13	9.1902785	0.6216423	1.6069190	2.3993107	3.0727266
14	9.1744842	0.6368822	1.6511227	2.4706666	3.1696366
15	9.1597610	0.6511227	1.6925154	2.5376134	3.2607171
16	9.1459727	0.6644866	1.7314334	2.6006651	3.3466317
17	9.1330077	0.6770758	1.7681562	2.6602508	3.4279367
18	9.1207733	0.6889750	1.8029183	2.7167322	3.5051026
19	9.1091914	0.7002560	1.8359186	2.7704171	3.5785315
20	9.0981960	0.7109799	1.8673271	2.8215696	3.6485694
21	9.0877306	0.7211990	1.8972903	2.8704181	3.7155162
22	9.0777464	0.7309589	1.9259355	2.9171615	3.7796337
23	9.0682010	0.7402989	1.9533737	2.9619739	3.8411517
24	9.0590577	0.7492537	1.9797027	3.0050085	3.9002732
25	9.0502837	0.7578539	2.0050085	3.0464012	3.9571780
26	9.0418506	0.7661264	2.0293679	3.0862727	4.0120267
27	9.0337327	0.7740954	2.0528490	3.1247310	4.0649628
28	9.0259073	0.7817822	2.0755129	3.1618728	4.1161153
29	9.0183542	0.7892062	2.0974148	3.1977853	4.1656007
30	9.0110560	0.7963858	2.1186051	3.2325484	4.2135252

MEMOIR No. 39.

On the Lunar Inequalities Produced by the Motion of the Ecliptic.

(Annals of Mathematics, Vol. I, pp. 5–10, 25–31, 52–58, 1884.)

This subject has been treated by Hansen* and more recently by Sir G. B. Airy and Prof. J. C. Adams.† Hansen's discussion is accommodated to the peculiar system of coordinates he employs, and the two later writers do not consider the inequalities in longitude. Hence an investigation, giving the inequalities of the latitude and longitude, at first, in the literal form, may be of value. The procedures employed are very similar to those of Pontécoulant, and doubtless are not as direct as might be imagined. The paper was written as long ago as 1867.

I.

Expressed in the ordinary notation, when the coordinates are referred to fixed planes, the differential equations of motion are

$$\frac{d^2X}{dt^2}+\frac{\mu}{r^3}X=\frac{\partial R}{\partial X},$$
$$\frac{d^2Y}{dt^2}+\frac{\mu}{r^3}Y=\frac{\partial R}{\partial Y},$$
$$\frac{d^2Z}{dt^2}+\frac{\mu}{r^3}Z=\frac{\partial R}{\partial Z}.$$

Since the directions of the axes are arbitrary, let the axis of X be directed towards the ascending node of the moving ecliptic on the ecliptic of 1850; and let the axis of Z be perpendicular to the latter plane. Taking now another system of coordinates, x, y and z, such that the axis of x has the same direction as that of X, but the axis of z is perpendicular to the moving ecliptic, let $\pi(t-1850)$ be the inclination of the moving ecliptic to that of 1850; then, neglecting quantities of the order of π^2, these equations exist

$$X=x,$$
$$Y=y-\pi(t-1850)z,$$
$$Z=z+\pi(t-1850)y.$$

* Darlegung, etc., Art. 175–178.

† Monthly Notices, Vol. XLI, pp. 264, 375 and 385.

The differential equations of motion, expressed in terms of the second system of coordinates, are

$$\frac{d^2x}{dt^2} + \frac{\mu}{r^3}x = \frac{\partial R}{\partial x},$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{r^3}y = \frac{\partial R}{\partial y} + 2\pi\frac{dz}{dt},$$

$$\frac{d^2z}{dt^2} + \frac{\mu}{r^3}z = \frac{\partial R}{\partial z} - 2\pi\frac{dy}{dt}.$$

Denoting the true longitude of the moon by λ, from these may be derived the two

$$\frac{d^2r^2}{2dt^2} - \frac{\mu}{r} + \frac{\mu}{a} = 2\int d'R + r\frac{\partial R}{\partial r} + 2\pi\frac{ydz - zdy}{dt},$$

$$\frac{d\left[(r^2 - z^2)\frac{d\lambda}{dt}\right]}{dt} = \frac{\partial R}{\partial \lambda} + 2\pi\frac{xdz}{dt}.$$

In this discussion all terms involving the solar eccentricity and parallax will be neglected. Let ζ denote the moon's mean angular distance from a point 90° behind the ascending node of the moving ecliptic on that of 1850, or $\zeta = \varepsilon + nt - \Pi + 90°$. For simplicity, the semi-axis major of the lunar orbit will be made equal to unity, and, as usual in the lunar theory, m will be written for $\frac{n'}{n}$. Also let ϕ and τ denote, respectively, the true and mean angular distance of the moon from the sun.

With these restrictions and notation

$$2\int d'R + r\frac{\partial R}{\partial r} = 4R + 2m\int\frac{\partial R}{\partial \lambda}d\zeta,$$

$$R = \frac{m^2}{4}\left[3(r^2 - z^2)\cos 2\varphi + r^2 - 3z^2\right],$$

$$\frac{\partial R}{\partial \lambda} = -\tfrac{3}{2}m^2(r^2 - z^2)\sin 2\varphi.$$

If the symbol δ prefixed to any quantity denote that part of it, in its development in series, which is multiplied by the first power of π, the equations for determining δr, $\delta\lambda$ and δz are

$$\frac{d^2(r\delta r)}{d\zeta^2} + \frac{\mu r\delta r}{n^2r^3} = 4\delta R + 2m\int\delta.\frac{\partial R}{\partial \lambda}d\zeta + 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta},$$

$$r^2\frac{d.\delta\lambda}{d\zeta} + 2\frac{d\lambda}{d\zeta}(r\delta r - z\delta z) = \int\delta.\frac{\partial R}{\partial \lambda}d\zeta + 2\frac{\pi}{n}\int xdz,$$

$$\frac{d^2\delta z}{d\zeta^2} + \left(\frac{\mu}{n^2r^3} + m^2\right)\delta z = -2\frac{\pi}{n}\frac{dy}{d\zeta}.$$

In these equations terms multiplied by the square and higher powers of the inclination of the moon's orbit are neglected; and, since δr and $\delta\lambda$ are multiplied by the first power of this quantity, this involves the neglect of terms such as $z\delta r$ and $z\delta\lambda$. For the same reason all higher powers of z than the second have been omitted in R.

These equations suffice to determine the inequalities we seek; but, for a term of long period in $\delta\lambda$, it will be more commodious to employ another equation. We have

$$r\frac{d^2r}{d\zeta^2}-(r^2-z^2)\frac{d\lambda^2}{d\zeta^2}-\left(\frac{dz}{d\zeta}-\frac{z}{r}\frac{dr}{d\zeta}\right)^2+\frac{\mu}{n^2r}=r\frac{\partial R}{\partial r}+2\frac{\pi}{n}\frac{ydz-zdy}{d\zeta},$$

or

$$\frac{r^2-z^2}{r^2}\frac{d\lambda^2}{d\zeta^2}-\frac{1}{r}\frac{d^2r}{d\zeta^2}+\left(\frac{1}{r}\frac{dz}{d\zeta}-\frac{z}{r^2}\frac{dr}{d\zeta}\right)^2-\frac{\mu}{n^2r^3}=-2\frac{R}{r^2}-2\frac{\pi}{n}\frac{ydz-zdy}{r^2d\zeta}.$$

Taking the variation with respect to π, and then multiplying by r^2,

$$\left\{\begin{aligned}&2r^2\frac{d\lambda}{d\zeta}\frac{d\,.\,\delta\lambda}{d\zeta}-2\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{rd^2\delta r-d^2r\,.\,\delta r}{d\zeta^2}\\&+2\left(\frac{dz}{d\zeta}-\frac{z}{r}\frac{dr}{d\zeta}\right)\left(\frac{d\,.\,\delta z}{d\zeta}-\frac{dr\delta z}{rd\zeta}\right)+\frac{3\mu r\delta r}{n^2r^3}\end{aligned}\right\}=\left\{\begin{aligned}&-2\frac{\partial R}{\partial\lambda}\delta\lambda-2\frac{\partial R}{\partial z}\delta z\\&-2\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}\end{aligned}\right\}.$$

But

$$3\frac{d^2(r\delta r)}{d\zeta^2}+3\frac{\mu r\delta r}{n^2r^3}=12\delta R+6m\int\delta\,.\,\frac{\partial R}{\partial\lambda}d\zeta+6\frac{\pi}{n}\frac{ydz-zdy}{d\zeta};$$

subtracting this

$$r^2\frac{d\lambda}{d\zeta}\frac{d\,.\,\delta\lambda}{d\zeta}=\left\{\begin{aligned}&\frac{d\left[2d(r\delta r)-dr\delta r\right]}{d\zeta^2}+\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{dz}{d\zeta}\frac{d\,.\,\delta z}{d\zeta}+\frac{d(z\delta z)}{d\zeta}\frac{dr}{rd\zeta}\\&-\left(\frac{dr}{rd\zeta}\right)^2z\delta z-7\delta R+\frac{\partial R}{\partial r}\delta r-3m\int\delta\,.\,\frac{\partial R}{\partial\lambda}d\zeta-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}\end{aligned}\right\}.$$

In determining δz we shall stop at terms of the order of $m\dfrac{\pi}{n}$, and shall neglect all terms multiplied by powers of the lunar eccentricity e higher than the first. In $\delta\lambda$ we shall neglect e altogether; and, since the inequalities in the lunar parallax resulting from δr are insensible, δr will be determined only so far as it is necessary to the determination of $\delta\lambda$. Let ξ denote the moon's mean anomaly, and η its mean argument of latitude, or $\eta=\varepsilon+nt-\Omega$. In applying the last equation to determining the coefficient of sin $(\zeta-\eta)$ in $\delta\lambda$ to terms of the order of $\gamma\dfrac{\pi}{n}$ (where γ denotes the same function of the

inclination as it does in Pontécoulant's *Théorie Analytique*) it will be necessary to compute each member to terms of the order of $m^2\gamma\,\frac{\pi}{n}$. But $r\delta r$ is of the order of $\gamma\,\frac{\pi}{n}$, consequently $\frac{d^2(r\delta r)}{d\zeta^2}$, in the term which has $\zeta - \eta$ for its argument, is of the order of $m^4\gamma\,\frac{\pi}{n}$ and thus may be neglected; moreover, $\frac{dr}{d\zeta}$ is of the order of m^2, and hence $\frac{d\,(dr\,.\,\delta r)}{d\zeta^2}$ is of the order of $m^4\gamma\,\frac{\pi}{n}$ in the term having the same argument; this may then also be omitted.

With these simplifications the last equation becomes

$$\begin{aligned} r^2\frac{d\lambda}{d\zeta}\frac{d\delta\lambda}{d\zeta} = \frac{d\lambda^2}{d\zeta^2}z\delta z - \frac{dz}{d\zeta}\frac{d\,.\,\delta z}{d\zeta} + \frac{dr}{rd\zeta}\frac{d\,(z\delta z)}{d\zeta} - \left(\frac{dr}{rd\zeta}\right)^2 z\delta z \\ + \tfrac{21}{2}\,m^2\,(1 + \cos 2\varphi)\,z\delta z - 4\,\frac{\pi}{n}\,\frac{ydz - zdy}{d\zeta} \\ - 3m^2\,(1 + 3\cos 2\varphi)\,r\delta r - 7\,\frac{\partial R}{\partial\lambda}\,\delta\lambda - 3m\int\delta\,.\,\frac{\partial R}{\partial\lambda}\,d\zeta\,. \end{aligned}$$

If, for brevity, we write

$$\begin{aligned} A &= \frac{\mu}{n^2r^3} + m^2\,, \\ B &= \frac{\mu}{n^2r^3} - 2m^2 - 6m^2\cos 2\varphi\,, \\ C &= 6m^2r^2\sin 2\varphi\,, \\ D &= 3m^2r^2\cos 2\varphi\,, \\ E &= 3m^2\sin 2\varphi\,, \\ U &= -\,2\,\frac{\pi}{n}\,\frac{dy}{d\zeta}, \\ U' &= 2\,\frac{\pi}{n}\,\frac{ydz - zdy}{d\zeta} - 6m^2\,(1 + \cos 2\varphi)\,z\delta z + 6m^2\int\sin 2\varphi\,.\,z\delta z\,d\zeta, \\ U'' &= 2\,\frac{\pi}{n}\int xdz + 2\,\frac{d\lambda}{d\zeta}\,z\delta z + 3m^2\int\sin 2\varphi\,.\,z\delta z d\zeta, \end{aligned}$$

the term $2m\int Er\delta rd\zeta$ in the equation for $r\delta r$ being omitted as not giving any terms which we wish to preserve, and it being sufficient to put $B = 1$, and $\frac{d\lambda}{d\zeta} = 1$, where the latter multiplies $r\delta r$ in the equation for $\delta\lambda$, the three equations become

$$\begin{aligned} \frac{d^2\delta z}{d\zeta^2} + A\delta z &= U\,, \\ \frac{d^2\,(r\delta r)}{d\zeta^2} + r\delta r + C\delta\lambda + 2m\int D\delta\lambda d\zeta &= U'\,, \\ r^2\,\frac{d\,.\,\delta\lambda}{d\zeta} + 2r\delta r + \int\,[D\delta\lambda + Er\delta r]\,d\zeta &= U''. \end{aligned}$$

To the degree of approximation we desire,

$$\frac{1}{r} = 1 + \tfrac{1}{6}m^2 + (m^2 + \tfrac{19}{6}m^3)\cos 2\tau,$$

$$\lambda = \epsilon + nt + (\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3)\sin 2\tau.$$

Also (Pontécoulant, *Théorie Analytique*, Tom. IV, pp. 216, 226)

$$A = 1 + \tfrac{3}{2}m^2 - \tfrac{9}{32}m^4 + \tfrac{55}{16}m^5 + (3m^2 + \tfrac{19}{2}m^3 + \tfrac{187}{6}m^4)\cos 2\tau + (3 + \tfrac{3}{2}m^2)e\cos\xi$$
$$+ (\tfrac{45}{8}m + \tfrac{657}{32}m^2)e\cos(2\tau - \xi) + \tfrac{147}{16}m^2 e\cos(2\tau + \xi).$$

From $y = r\sin(\lambda - \Pi)$ we derive

$$y = -\left(1 - \frac{m^2}{6}\right)\cos\zeta + \tfrac{19}{16}m^2\cos(\zeta - 2\tau) - \tfrac{1}{2}e\cos(\zeta + \xi),$$

$$U = -\left(2 - \frac{m^2}{3}\right)\frac{\pi}{n}\sin\zeta - \tfrac{19}{8}m^2\frac{\pi}{n}\sin(\zeta - 2\tau) - 2e\frac{\pi}{n}\sin(\zeta + \xi).$$

Let

$$\delta z = \frac{\pi}{n}\Big\{A_1\sin\zeta + A_2\sin(\zeta - 2\tau) + A_3\sin(\zeta + 2\tau) + A_4\sin(\zeta - 4\tau)$$
$$+ A_5 e\sin(\zeta - \xi) + A_6 e\sin(\zeta + \xi) + A_7 e\sin(\zeta - 2\tau + \xi)$$
$$+ A_8 e\sin(\zeta + 2\tau - \xi) + A_9 e\sin(\zeta - 2\tau - \xi) + A_{10} e\sin(\zeta + 2\tau + \xi)$$
$$+ A_{11} e\sin(\zeta - 4\tau + \xi)\Big\}.$$

On substituting this expression in the first of the three differential equations, the following equations result for determining A_1, A_2, etc.,

$$\begin{array}{lcl}
(\tfrac{3}{2}m^2 - \tfrac{9}{32}m^4 + \tfrac{55}{16}m^5)A_1 + (\tfrac{3}{2}m^2 + \tfrac{19}{4}m^3 + \tfrac{187}{12}m^4)(A_2 + A_3) & = & -2 + \tfrac{1}{3}m^2, \\
(4m - \tfrac{5}{2}m^2)A_2 + (\tfrac{3}{2}m^2 + \tfrac{19}{4}m^3 + \tfrac{187}{12}m^4)A_1 & = & -\tfrac{19}{8}m^2, \\
-(8 - 12m)A_3 + (\tfrac{3}{2}m^2 + \tfrac{19}{4}m^3)A_1 & = & 0, \\
-8A_4 + \tfrac{3}{2}m^2 A_2 & = & 0, \\
(1 + \tfrac{3}{2}m^2)A_5 + (\tfrac{3}{2} + \tfrac{3}{4}m)A_1 + \tfrac{45}{16}mA_2 & = & 0, \\
-(3 - \tfrac{3}{2}m^2)A_6 + (\tfrac{3}{2} + \tfrac{3}{4}m)A_1 & = & -2, \\
A_7 + \tfrac{3}{2}m^2 A_6 + \tfrac{3}{2}A_2 + (\tfrac{45}{16}m + \tfrac{657}{64}m^2)A_1 & = & 0, \\
-(3 - 8m)A_8 + \tfrac{3}{2}m^2 A_5 + \tfrac{3}{2}A_3 + (\tfrac{45}{16}m + \tfrac{657}{64}m^2)A_1 & = & 0, \\
-(3 - 8m)A_9 + \tfrac{3}{2}m^2 A_5 + \tfrac{3}{2}A_2 + \tfrac{147}{32}m^2 A_1 & = & 0, \\
-15A_{10} + \tfrac{3}{2}m^2 A_6 + \tfrac{3}{2}A_3 + \tfrac{147}{32}m^2 A_1 & = & 0, \\
-3A_{11} + \tfrac{45}{16}mA_3 & = & 0.
\end{array}$$

By the solution of these, this expression of δz is obtained

$$\begin{aligned}
\delta z = &-\left(\tfrac{4}{3}m^{-2} + \tfrac{1}{2}m^{-1} + \tfrac{21}{8} + \tfrac{3395}{288}m\right)\frac{\pi}{n}\sin\zeta \\
&+\left(\tfrac{1}{2}m^{-1} + \tfrac{25}{12} + \tfrac{1843}{288}m\right)\frac{\pi}{n}\sin(\zeta - 2\tau) \\
&-\left(\tfrac{1}{4} + \tfrac{131}{96}m\right)\frac{\pi}{n}\sin(\zeta + 2\tau) \\
&+\tfrac{3}{32}m\frac{\pi}{n}\sin(\zeta - 4\tau) \\
&+\left(2m^{-2} + \tfrac{3}{4}m^{-1} + \tfrac{169}{96}\right)e\frac{\pi}{n}\sin(\zeta - \xi) \\
&-\left(\tfrac{3}{2}m^{-2} + \tfrac{1}{4}m^{-1} + \tfrac{43}{16}\right)e\frac{\pi}{n}\sin(\zeta + \xi) \\
&+\left(3m^{-1} + \tfrac{415}{32}\right)e\frac{\pi}{n}\sin(\zeta - 2\tau + \xi) \\
&-\left(\tfrac{5}{4}m^{-1} + \tfrac{719}{96}\right)e\frac{\pi}{n}\sin(\zeta + 2\tau - \xi) \\
&+\left(\tfrac{1}{4}m^{-1} - \tfrac{3}{32}\right)e\frac{\pi}{n}\sin(\zeta - 2\tau - \xi) \\
&-\tfrac{1}{2}e\frac{\pi}{n}\sin(\zeta + 2\tau + \xi) + \tfrac{15}{32}e\frac{\pi}{n}\sin(\zeta - 4\tau + \xi).
\end{aligned}$$

The value of z (*Théorie Analytique*, Tom. IV, pp. 237, 244) is

$$\begin{aligned}
z = \gamma\Big\{&\left(1 - \frac{m^2}{6} + \tfrac{57}{128}m^3\right)\sin\eta + \left(\tfrac{3}{8}m + \tfrac{41}{32}m^2 + \tfrac{5293}{1536}m^3\right)\sin(2\tau - \eta) \\
&+ \left(\tfrac{3}{16}m^2 + \tfrac{7}{8}m^3\right)\sin(2\tau + \eta)\Big\},
\end{aligned}$$

whence, by multiplication is obtained

$$\begin{aligned}
z\delta z = &-\left(\tfrac{2}{3}m^{-2} + \tfrac{1}{4}m^{-1} + \tfrac{421}{288}\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta) \\
&+\left(\tfrac{1}{4}m^{-1} + \tfrac{11}{12} + \tfrac{21}{16}m\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta - 2\tau) \\
&+\left(\tfrac{1}{4}m^{-1} + \tfrac{79}{16} + \tfrac{9987}{2304}m\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta + 2\tau) \\
&+\left(\tfrac{2}{3}m^{-2} + \tfrac{1}{4}m^{-1} + \tfrac{29}{16}\right)\gamma\frac{\pi}{n}\cos(\zeta + \eta) \\
&-\left(\tfrac{1}{2}m^{-1} + \tfrac{191}{96} + \tfrac{14795}{2304}m\right)\gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau) \\
&+\tfrac{1}{4}\gamma\frac{\pi}{n}\cos(\zeta + \eta + 2\tau) + \tfrac{3}{32}\gamma\frac{\pi}{n}\cos(\zeta + \eta - 4\tau).
\end{aligned}$$

Also we get

$$2\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}=-\left(2+\tfrac{1}{12}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta)-\tfrac{3}{4}m\gamma\frac{\pi}{n}\cos(\zeta+\eta-2\tau),$$

$$\begin{aligned}-6m^2(1+\cos 2\tau)z\delta z=\ &4\gamma\frac{\pi}{n}\cos(\zeta-\eta)+\left(2-\tfrac{3}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta-2\tau)\\ &-4\gamma\frac{\pi}{n}\cos(\zeta+\eta)+\left(2-\tfrac{3}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta+2\tau)\\ &-\left(2-\tfrac{9}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta+\eta-2\tau)-2\gamma\frac{\pi}{n}\cos(\zeta+\eta+2\tau),\end{aligned}$$

$$\begin{aligned}6m^2\int z\delta z\sin 2\tau\,.\,d\zeta=\ &m\gamma\frac{\pi}{n}\cos(\zeta-\eta-2\tau)+m\gamma\frac{\pi}{n}\cos(\zeta-\eta+2\tau)\\ &+\gamma\frac{\pi}{n}\cos(\zeta+\eta-2\tau),\end{aligned}$$

and, by the addition of these three equations,

$$\begin{aligned}U'=\gamma\frac{\pi}{n}\Big\{&2\cos(\zeta-\eta)+\left(2+\tfrac{1}{4}m\right)\cos(\zeta-\eta-2\tau)+\left(2+\tfrac{1}{4}m\right)\cos(\zeta-\eta+2\tau)\\ &-4\cos(\zeta+\eta)-\left(1-\tfrac{3}{2}m\right)\cos(\zeta+\eta-2\tau)-2\cos(\zeta+\eta+2\tau)\Big\}.\end{aligned}$$

In the next place

$$2\frac{\pi}{n}x\frac{dz}{d\zeta}=\gamma\frac{\pi}{n}\left\{\tfrac{3}{8}m\sin(\zeta-\eta+2\tau)+\sin(\zeta+\eta)+\left(\tfrac{3}{8}m-\tfrac{21}{32}m^2\right)\sin(\zeta+\eta-2\tau)\right\}$$

$$\begin{aligned}3m^2z\delta z\sin 2\tau=\gamma\frac{\pi}{n}\Big\{&\left(1+\tfrac{3}{8}m\right)\sin(\zeta-\eta-2\tau)-\left(1+\tfrac{3}{8}m\right)\sin(\zeta-\eta+2\tau)\\ &-\left(1+\tfrac{3}{8}m\right)\sin(\zeta+\eta-2\tau)+\sin(\zeta+\eta+2\tau)\Big\},\end{aligned}$$

$$\begin{aligned}2\frac{d\lambda}{d\zeta}z\delta z=\gamma\frac{\pi}{n}\Big\{&\left(\tfrac{1}{2}m^{-1}-\tfrac{17}{48}m\right)\cos(\zeta-\eta-2\tau)+\left(\tfrac{1}{2}m^{-1}-\tfrac{3}{16}-\tfrac{55}{384}m\right)\cos(\zeta-\eta+2\tau)\\ &+\left(\tfrac{4}{3}m^{-2}+\tfrac{1}{2}m^{-1}+\tfrac{29}{9}\right)\cos(\zeta+\eta)-\left(m^{-1}+\tfrac{103}{48}+\tfrac{8563}{1152}m\right)\cos(\zeta+\eta-2\tau)\\ &+\tfrac{7}{8}\cos(\zeta+\eta+2\tau)+\tfrac{3}{16}\cos(\zeta+\eta-4\tau)\Big\}.\end{aligned}$$

In these expressions the terms depending on the argument $\zeta-\eta$ are omitted because the coefficient belonging to this argument in $\delta\lambda$ will be determined from the differential equation given specially for this purpose.

Remembering that

$$\frac{d\eta}{d\zeta}=1+\tfrac{3}{4}m^2-\tfrac{9}{32}m^3-\tfrac{273}{128}m^4,$$

the following expression for U'' is readily obtained:

$$\begin{aligned}U''=\gamma\frac{\pi}{n}\Big\{&\left(\tfrac{1}{2}m^{-1}+\tfrac{1}{2}+\tfrac{1}{8}m\right)\cos(\zeta-\eta-0\tau)\\ &+\left(\tfrac{1}{2}m^{-1}+\tfrac{5}{16}+\tfrac{187}{384}m\right)\cos(\zeta-\eta+2\tau)\\ &+\left(\tfrac{4}{3}m^{-2}+\tfrac{1}{2}m^{-1}+\tfrac{4}{15}\right)\cos(\zeta+\eta)\\ &-\left(\tfrac{1}{2}m^{-1}+\tfrac{7}{8}+\tfrac{883}{144}m\right)\cos(\zeta+\eta-2\tau)\\ &+\tfrac{25}{12}\cos(\zeta+\eta+2\tau)+\tfrac{3}{16}\cos(\zeta+\eta-4\tau)\Big\}.\end{aligned}$$

Let us now put

$$
\begin{aligned}
r\delta r = \gamma \frac{\pi}{n} \Big\{ & B_1 \cos(\zeta - \eta) + B_2 \cos(\zeta - \eta - 2\tau) \\
& + B_3 \cos(\zeta - \eta + 2\tau) + B_4 \cos(\zeta + \eta) \\
& + B_5 \cos(\zeta + \eta - 2\tau) + B_6 \cos(\zeta + \eta + 2\tau) \Big\}, \\
\delta\lambda = \gamma \frac{\pi}{n} \Big\{ & C_1 \sin(\zeta - \eta) + C_2 \sin(\zeta - \eta - 2\tau) \\
& + C_3 \sin(\zeta - \eta + 2\tau) + C_4 \sin(\zeta + \eta) + C_5 \sin(\zeta + \eta - 2\tau) \\
& + C_6 \sin(\zeta + \eta + 2\tau) + C_7 \sin(\zeta + \eta - 4\tau) \Big\}.
\end{aligned}
$$

To a sufficient degree of approximation

$$
\begin{aligned}
C &= 6m^2 \sin 2\tau, \\
D &= -\tfrac{57}{8} m^4 - \tfrac{97}{4} m^5 + (3m^2 - m^4)\cos 2\tau, \\
E &= 3m^2 \sin 2\tau.
\end{aligned}
$$

Substituting the expressions for $r\delta r$ and $\delta\lambda$ in the differential equations which serve to determine them, the following equations of condition between the coefficients are obtained:

$$
\begin{aligned}
B_1 &= 2, \\
-(3 - 8m) B_2 + (3^2 + m\tfrac{3}{2}^3)\, m C_1 &= 2 + \tfrac14 m, \\
-(3 - 8m) B_3 - (3m^2 + \tfrac{3}{2} m^3) C_1 &= 2 + \tfrac14 m, \\
3B_4 &= 4, \\
-B_5 - (\tfrac{3}{2} m^2 + \tfrac{9}{16} m^3) C_4 &= 1 - \tfrac{3}{2} m, \\
15B_6 + 3m^2 C_4 &= 2, \\
(2 - 2m + \tfrac{1}{12} m^2) C_2 - (\tfrac34 m^2 + \tfrac34 m^3) C_1 - 2B_2 &= -\tfrac12 m^{-1} - \tfrac12 - \tfrac18 m, \\
(2 - 2m - \tfrac{17}{12} m^2) C_3 - (\tfrac34 m^2 + \tfrac34 m^3) C_1 + 2B_3 &= \tfrac12 m^{-1} + \tfrac{5}{16} + \tfrac{187}{384} m, \\
(2 + \tfrac{1}{12} m^2) C_4 + 2B_4 &= \tfrac43 m^{-2} + \tfrac12 m^{-1} + \tfrac{49}{18}, \\
\left\{ \begin{array}{c} (2m + \tfrac34 m^2) C_5 - (\tfrac34 m + \tfrac{55}{32} m^2 + \tfrac{2417}{384} m^3) C_4 \\ - \tfrac34 m C_7 + 2B_5 + \tfrac34 m B_4 \end{array} \right\} &= -\tfrac12 m^{-1} - \tfrac78 - \tfrac{889}{144} m, \\
4C_6 - 2m^2 C_4 + 2B_6 &= \tfrac{25}{12}, \\
2C_7 &= -\tfrac{3}{16}.
\end{aligned}
$$

To obtain an equation for determining C_1 we employ the special differential equation we have given for this purpose. Here we have

$$
\begin{aligned}
\frac{d\lambda^2}{d\zeta^2} &= 1 + \tfrac{121}{32} m^4 + (\tfrac{11}{2} m^2 + \tfrac{85}{6} m^3) \cos 2\tau, \\
-\left(\frac{dr}{r\,d\zeta}\right)^2 &= -2m^4, \\
\tfrac{21}{2} m^2 (1 + \cos 2\varphi) &= \tfrac{21}{2} m^2 - \tfrac{231}{16} m^4 + \tfrac{21}{2} m^2 \cos 2\tau, \\
\frac{d\lambda^2}{d\zeta^2} - \left(\frac{dr}{r\,d\zeta}\right)^2 + \tfrac{21}{2} m^2 (1 + \cos 2\varphi) &= 1 + \tfrac{21}{2} m^2 - \tfrac{405}{32} m^4 + (16m^2 + \tfrac{85}{6} m^3) \cos 2\tau.
\end{aligned}
$$

Retaining only the term whose argument is $\zeta - \eta$,

$$\left\{ \frac{d\lambda^2}{d\zeta^2} - 1 - \left(\frac{dr}{rd\zeta}\right)^2 + \tfrac{21}{2} m^2 (1 + \cos 2\varphi) \right\} z\delta z$$
$$= -\left(7 - \tfrac{11}{8} m - \tfrac{15835}{192} m^2\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta).$$

In addition,

$$\frac{dr}{rd\zeta} = \left(2m^2 + \tfrac{13}{8} m^3\right) \sin 2\tau,$$
$$\frac{dr}{rd\zeta} \frac{d(z\delta z)}{d\zeta} = -\left(m + \tfrac{223}{48} m^2\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta),$$
$$-4 \frac{\pi}{n} \frac{ydz - zdy}{d\zeta} = \left(4 + \tfrac{1}{6} m^2\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta).$$

Let us write the series for z

$$z = \gamma \{ q_1 \sin \eta + q_2 \sin(2\tau - \eta) + q_3 \sin(2\tau + \eta) \},$$

then

$$z\delta z = \tfrac{1}{2} (A_1 q_1 - A_2 q_2 + A_3 q_3) \gamma \frac{\pi}{n} \cos(\zeta - \eta),$$
$$\frac{dz}{d\zeta} = \gamma \left\{ \left(1 + \tfrac{3}{4} m^2 - \tfrac{9}{32} m^3 - \tfrac{273}{128} m^4\right) q_1 \cos \eta \right.$$
$$\left. + \left(1 - 2m - \tfrac{3}{4} m^2\right) q_2 \cos(2\tau - \eta) + 3 q_3 \cos(2\tau + \eta) \right\},$$
$$\frac{d\delta z}{d\zeta} = \frac{\pi}{n} \left\{ A_1 \cos \zeta - (1 - 2m) A_2 \cos(\zeta - 2\tau) + 3 A_3 \cos(\zeta + 2\tau) \right\},$$
$$z\delta z - \frac{dz}{d\zeta} \frac{d.\delta z}{d\zeta} = -\tfrac{1}{2} \left\{ \left(\tfrac{3}{4} m^2 - \tfrac{9}{32} m^3 - \tfrac{273}{128} m^4\right) A_1 q_1 \right.$$
$$\left. + \left(4m - \tfrac{13}{4} m^2\right) A_2 q_2 + 8 A_3 q_3 \right\} \gamma \frac{\pi}{n} \cos(\zeta - \eta).$$

Substituting in the last equation the values of A_1, q_1 . . . . , it becomes

$$z\delta z - \frac{dz}{d\zeta} \frac{d.\delta z}{d\zeta} = \left(\tfrac{1}{2} - \tfrac{3}{8} m - \tfrac{253}{96} m^2\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta).$$

Also we have

$$-3m^2 (1 + 3 \cos 2\tau) r\delta r = \left[3m^2 B_1 + \tfrac{9}{2} m^2 (B_2 + B_3)\right] \gamma \frac{\pi}{n} \cos(\zeta - \eta);$$

but, from the previous equations of condition, $B_1 = 2$, and $B_2 + B_3 = -\tfrac{4}{3}$, hence

$$-3m^2 (1 + 3 \cos 2\tau) r\delta r = 0.$$

In addition

$$\frac{\partial R}{\partial \lambda} = -\tfrac{3}{2} m^2 \sin 2\tau,$$

$$-\gamma \frac{\partial R}{\partial \lambda} \delta\lambda = -\tfrac{21}{4} m^2 (C_2 - C_3) \gamma \frac{\pi}{n} \cos(\zeta - \eta),$$

$$r^2 \frac{d\lambda}{d\zeta} = 1 - \tfrac{1}{2} m^2 + (\tfrac{5}{4} m^2 + \tfrac{4}{3} m^3) \cos 2\tau,$$

$$-3m \int \delta . \frac{\partial R}{\partial \lambda} d\zeta = 3m \int [D\delta\lambda + E(r\delta r - z\delta z)]\, d\zeta,$$

$$3m \int D\delta\lambda d\zeta = -\left\{ (\tfrac{57}{2} m^3 + \tfrac{1723}{16} m^4) C_1 - (6m + \tfrac{9}{4} m^2)(C_2 + C_3) \right\} \gamma \frac{\pi}{n} \cos(\zeta - \eta),$$

$$3m \int E r\delta r d\zeta = (6m + \tfrac{9}{4} m^2)(B_2 - B_3) \gamma \frac{\pi}{n} \cos(\zeta - \eta),$$

$$-3m \int E z\delta z d\zeta = -(\tfrac{9}{16} m - \tfrac{27}{64} m^2) \gamma \frac{\pi}{n} \cos(\zeta - \eta).$$

Thus is obtained the equation which determines C_1:

$$\left\{ \begin{array}{l} (\tfrac{5}{4} m^2 - \tfrac{9}{32} m^3 - \tfrac{305}{128} m^4) C_1 - \tfrac{9}{2} m^2 (C_2 - C_3) \\ -(\tfrac{57}{2} m^3 + \tfrac{1723}{16} m^4) C_1 + (6m + \tfrac{9}{4} m^2)(B_2 - B_3 + C_2 + C_3) \end{array} \right\} = \tfrac{5}{2} + \tfrac{9}{16} m - \tfrac{125}{96} m^2.$$

But the previous equations of condition furnish

$$C_2 - C_3 = -\tfrac{1}{2} m^{-1} - \tfrac{215}{96},$$
$$B_2 - B_3 + C_2 + C_3 = (\tfrac{19}{4} m^2 + \tfrac{27}{8} m^3) C_1 - \tfrac{3}{32} + \tfrac{27}{256} m,$$

consequently

$$(\tfrac{5}{4} m^2 - \tfrac{9}{32} m^3 - \tfrac{305}{128} m^4) C_1 = \tfrac{5}{2} - \tfrac{9}{8} m - \tfrac{1182}{96} m^2,$$

and

$$C_1 = \tfrac{10}{5} m^{-2} - \tfrac{1}{4} m^{-1} - \tfrac{502}{96}.$$

Solving the remaining equations of condition we get

$$\begin{aligned} \delta\lambda = {} & (\tfrac{10}{5} m^{-2} - \tfrac{1}{4} m^{-1} - \tfrac{502}{96}) \gamma \frac{\pi}{n} \sin(\zeta - \eta) \\ & - (\tfrac{1}{4} m^{-1} - \tfrac{41}{12}) \gamma \frac{\pi}{n} \sin(\zeta - \eta - 2\tau) \\ & + (\tfrac{1}{4} m^{-1} + \tfrac{181}{32}) \gamma \frac{\pi}{n} (\zeta - \eta + 2\tau) \\ & + (\tfrac{2}{3} m^{-2} + \tfrac{1}{4} m^{-1} + 0) \gamma \frac{\pi}{n} \sin(\zeta + \eta) \\ & + (\tfrac{3}{2} m^{-1} - \tfrac{285}{96}) \gamma \frac{\pi}{n} \sin(\zeta + \eta - 2\tau) \\ & + \tfrac{41}{48} \gamma \frac{\pi}{n} \sin(\zeta + \eta + 2\tau) - \tfrac{3}{32} \gamma \frac{\pi}{n} \sin(\zeta + \eta - 4\tau). \end{aligned}$$

The expression for the inequalities in latitude is

$$
\begin{aligned}
\delta\beta = \frac{\delta z}{r} = & - \left(\tfrac{4}{3} m^{-2} + \tfrac{1}{2} m^{-1} + \tfrac{11}{8} + \tfrac{3847}{288} m\right) \frac{\pi}{n} \sin \zeta \\
& + \left(\tfrac{1}{2} m^{-1} + \tfrac{17}{12} + \tfrac{1187}{288} m\right) \frac{\pi}{n} \sin (\zeta - 2\tau) \\
& - \left(\tfrac{11}{12} + \tfrac{1043}{288} m\right) \frac{\pi}{n} \sin (\zeta + 2\tau) + \tfrac{11}{32} m \frac{\pi}{n} \sin (\zeta - 4\tau) \\
& + \left(\tfrac{4}{3} m^{-2} + \tfrac{1}{2} m^{-1} + \tfrac{35}{48}\right) e \frac{\pi}{n} \sin (\zeta - \xi) \\
& - \left(\tfrac{4}{3} m^{-2} + \tfrac{1}{2} m^{-1} + \tfrac{13}{8}\right) e \frac{\pi}{n} \sin (\zeta + \xi) \\
& + (2 m^{-1} + 2)\, e \frac{\pi}{n} \sin (\zeta - 2\tau + \xi) - \left(\tfrac{5}{2} m^{-1} + \tfrac{527}{48}\right) e \frac{\pi}{n} \sin (\zeta + 2\tau - \xi) \\
& + \left(\tfrac{1}{2} m^{-1} + \tfrac{7}{24}\right) e \frac{\pi}{n} \sin (\zeta - 2\tau - \xi) - \tfrac{7}{8} e \frac{\pi}{n} \sin (\zeta + 2\tau + \xi) \\
& + \tfrac{15}{16} e \frac{\pi}{n} \sin (\zeta - 4\tau + \xi).
\end{aligned}
$$

II.

The direct action of the planets produces in the motion of the moon terms which have nearly the same periods as those we have been considering. To complete the subject it is necessary to derive these and add them to those just obtained. If $\delta' R$ denote the part of R which is due to the action of the planet m'', and Δ the distance of the latter from the earth, two accents being used to denote quantities which belong to the planet,

$$
\delta' R = m'' \left\{ [\Delta^2 - 2(x'' x + y'' y + z'' z) + r^2]^{-\frac{1}{2}} - \frac{x'' x + y'' y + z'' z}{r''^3} \right\}.
$$

Or with sufficient approximation,

$$
\delta' R = \frac{m''}{2} \left\{ 3 \frac{(x'' x + y'' y + z'' z)^2}{\Delta^5} - \frac{r^2}{\Delta^3} \right\}.
$$

The only part of $\delta' R$ which can produce terms we are in search of is that which has z'' for a factor; thus we may take

$$
\delta' R = 3 m'' \frac{(x'' x + y'' y) z'' z}{\Delta^5}.
$$

But, with sufficient approximation

$$
\begin{aligned}
x'' &= a'' \cos (\epsilon'' + n'' t) + a' \cos (\epsilon' + n' t), \\
y'' &= a'' \sin (\epsilon'' + n'' t) + a' \sin (\epsilon' + n' t), \\
z'' &= a'' \gamma'' \sin (\epsilon'' + n'' t - \Omega''), \\
\Delta^{-5} &= \tfrac{1}{2} A_0 + A_1 \cos [\epsilon'' - \epsilon' + (n'' - n') t] + \dots, \\
x'' x + y'' y &= a'' r \cos (\lambda - \epsilon'' - n'' t) + a' r \cos (\lambda - \epsilon' - n' t).
\end{aligned}
$$

Preserving only terms which are needed,

$$(x'' x + y'' y)\, z'' = \tfrac{1}{2} a'' \gamma'' r \{ a'' \sin (\lambda - \Omega'') \\ \qquad + a' \sin [\lambda - \Omega'' + \epsilon'' - \epsilon' + (n'' - n') t] \},$$

$$\frac{(x'' x + y'' y)\, z'' z}{\Delta^5} = \tfrac{1}{4} a'' \gamma'' (a'' A_0 + a' A_1) r z \sin (\lambda - \Omega'').$$

Consequently

$$\begin{aligned} \delta' R &= \tfrac{3}{4} m'' a'' \gamma'' (a'' A_0 + a' A_1) r z \sin (\lambda - \Omega'') \\ &= \tfrac{3}{4} \frac{m''}{m'} m^2 a'^3 a'' \gamma'' (a'' A_0 + a' A_1) r z \sin (\lambda - \Omega'') \\ &= - 2 K r z \sin (\lambda - \Omega''). \end{aligned}$$

For an inferior planet

$$A_0 = a'^{-5} b_{\frac{5}{2}}^{(0)}, \qquad\qquad A_1 = - a'^{-5} b_{\frac{5}{2}}^{(1)},$$

and for a superior one

$$A_0 = a''^{-5} b_{\frac{5}{2}}^{(0)}, \qquad\qquad A_1 = - a''^{-5} b_{\frac{5}{2}}^{(1)}.$$

But

$$b_{\frac{5}{2}}^{(0)} = \frac{(1 + a^2)\, b_{\frac{3}{2}}^{(0)} + \frac{2}{3} a\, b_{\frac{3}{2}}^{(1)}}{(1 - a^2)^2}, \qquad\qquad b_{\frac{5}{2}}^{(1)} = \frac{2\, a\, b_{\frac{3}{2}}^{(0)} + \frac{1}{3} (1 + a^2)\, b_{\frac{3}{2}}^{(1)}}{(1 - a^2)^2}$$

Consequently, for an inferior planet,

$$K = \tfrac{3}{8} \frac{m''}{m'} m^2 \gamma'' \frac{a}{1 - a^2} \left(a b_{\frac{3}{2}}^{(0)} + \tfrac{1}{3} b_{\frac{3}{2}}^{(1)}\right),$$

and, for a superior,

$$K = - \tfrac{3}{8} \frac{m''}{m'} m^2 \gamma'' \frac{a^3}{1 - a^2} \left(b_{\frac{3}{2}}^{(0)} + \tfrac{1}{3} a b_{\frac{3}{2}}^{(1)}\right).$$

To determine δz we shall have the equation

$$\frac{d^2\, \delta z}{d\zeta^2} + A\, \delta z = - 2\, K r \sin (\lambda - \Omega'').$$

Making $\varepsilon + nt - \Omega'' = \zeta'$,

$$K r \sin (\lambda - \Omega'') = \left(1 - \frac{m^2}{6}\right) K \sin \zeta' - \tfrac{19}{16} m^2 K \sin (\zeta' - 2\tau).$$

Since K is much smaller than $\dfrac{\pi}{n}$, we shall content ourselves with one order of approximation less in the factors which multiply it than in those which multiply $\dfrac{\pi}{n}$. With this restriction it will be readily seen that the value of δz is obtained simply by writing K and ζ' for $\dfrac{\pi}{n}$ and ζ in the formula previously obtained. Thus

$$\delta z = - \left(\tfrac{4}{3} m^{-2} + \tfrac{1}{2} m^{-1} + \tfrac{21}{8}\right) K \sin \zeta' \\ \qquad + \left(\tfrac{1}{2} m^{-1} + \tfrac{25}{12}\right) K \sin (\zeta' - 2\tau) - \tfrac{1}{4} K \sin (\zeta' + 2\tau).$$

As regards the differential equations which determine $r\delta r$ and $\delta\lambda$, it is evident that they remain the same as before, with the exceptions that K and ζ' everywhere take the place of $\frac{\pi}{n}$ and ζ; and in U', in place of $2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta}$, must be put $4\delta' R = 0$, and that, consequently, U' in this case becomes

$$U' = \gamma K \{2 \cos(\zeta' - \eta - 2\tau) + 2 \cos(\zeta' - \eta + 2\tau) - \cos(\zeta' + \eta - 2\tau)\};$$

and in U'' in place of $2\frac{\pi}{n}\int xdz$, must be put

$$-2K\int rz \cos(\lambda - \Omega'')\, d\zeta = -\tfrac{3}{16}\gamma K \cos(\zeta' + \eta - 2\tau),$$

whence U'' in this case becomes

$$U'' = \gamma K \{(\tfrac{1}{2} m^{-1} + \tfrac{1}{2}) \cos(\zeta' - \eta - 2\tau) + (\tfrac{1}{2} m^{-1} + \tfrac{5}{16}) \cos(\zeta' - \eta + 2\tau)$$
$$+ (\tfrac{3}{8} m^{-2} + \tfrac{1}{2} m^{-1}) \cos(\zeta' + \eta) - (\tfrac{1}{2} m^{-1} + \tfrac{7}{8}) \cos(\zeta' + \eta - 2\tau)\};$$

and, in the differential equation determining the coefficient of $\sin(\zeta - \eta)$ in $\delta\lambda$, in place of $-4\frac{\pi}{n}\frac{ydz - zdy}{d\zeta}$, must be put

$$-7\,\delta' R = 7\gamma K \cos(\zeta' - \eta).$$

Making use of similar expressions for $r\delta r$ and $\delta\lambda$ as were used in the former case, we obtain the equations of condition

$$-3B_2 + 3m^2 C_1 = 2,$$
$$-3B_3 - 3m^2 C_1 = 2,$$
$$-B_5 - \tfrac{3}{2} m^2 C_4 = 1,$$
$$\left\{\begin{array}{l}(\tfrac{3}{4} m^2 - \tfrac{9}{32} m^3) C_1 - \tfrac{9}{2} m^2 (C_2 - C_3) \\ -\tfrac{57}{2} m^3 C_1 + 6m(B_2 - B_3 + C_2 + C_3)\end{array}\right\} = -\tfrac{1}{2} + \tfrac{9}{16} m,$$
$$(2 - 2m) C_2 - \tfrac{3}{4} m^2 C_1 - 2B_2 = -\tfrac{1}{2} m^{-1} - \tfrac{1}{2},$$
$$(2 - 2m) C_3 - \tfrac{3}{4} m^2 C_1 + 2B_3 = \tfrac{1}{2} m^{-1} + \tfrac{5}{16},$$
$$2C_4 = \tfrac{3}{8} m^{-2} + \tfrac{1}{2} m^{-1},$$
$$2mC_5 - (\tfrac{3}{4} m + \tfrac{55}{32} m^2) C_4 + 2B_5 = -\tfrac{1}{2} m^{-1} - \tfrac{7}{8}.$$

These equations are the same as those we obtained in the case of the inequalities produced by the motion of the ecliptic, with the single exception of that which determines C_1; and, being solved, they give

$$\delta\lambda = -(\tfrac{3}{8} m^{-2} + \tfrac{3}{2} m^{-1})\gamma K \sin(\zeta' - \eta) - \tfrac{1}{4} m^{-1}\gamma K \sin(\zeta' - \eta - 2\tau)$$
$$+ \tfrac{1}{4} m^{-1}\gamma K \sin(\zeta' - \eta + 2\tau) + (\tfrac{3}{8} m^{-2} + \tfrac{1}{4} m^{-1})\gamma K \sin(\zeta' + \eta)$$
$$+ \tfrac{3}{2} m^{-1}\gamma K \sin(\zeta' + \eta - 2\tau).$$

The expression for the inequalities in latitude is

$$\delta\beta = -\left(\tfrac{3}{8} m^{-2} + \tfrac{1}{2} m^{-1} + \tfrac{11}{8}\right) K \sin \zeta' + \left(\tfrac{1}{2} m^{-1} + \tfrac{17}{12}\right) K \sin (\zeta' - 2\tau)$$
$$- \tfrac{11}{12} K \sin (\zeta' + 2\tau).$$

III.

It remains only to transform the foregoing formulas into numerical results. According to Hansen and Olufsen (*Tables du Soleil, Introduction*),

$$\pi \sin \Pi = +0''.053916, \qquad \pi \cos \Pi = -0''.467839,$$

whence

$$\pi = 0''.470903, \qquad \Pi = 173°25'34''.$$

Also

$$n = 17325225'', \quad m = 0.074801, \quad \gamma = 0.089673, \quad e = 0.054731.$$

Substituting these values, the inequalities produced by the motion of the ecliptic are

$$\begin{aligned}\delta\lambda = &+0''.2952 \sin (\zeta - \eta) + 0''.0000 \sin (\zeta - \eta - 2\tau) + 0''.0045 \sin (\zeta - \eta + 2\tau)\\ &+ 0''.0616 \sin (\zeta + \eta) + 0''.0089 \sin (\zeta + \eta - 2\tau) + 0''.0004 \sin (\zeta + \eta + 2\tau)\\ &+ 0''.0000 \sin (\zeta + \eta - 4\tau),\\ \delta\beta = &-1''.4001 \sin \zeta + 0''.0469 \sin (\zeta - 2\tau) - 0''.0064 \sin (\zeta + 2\tau)\\ &+ 0''.0001 \sin (\zeta - 4\tau) + 0''.0757 \sin (\zeta - \xi) - 0''.0768 \sin (\zeta + \xi)\\ &+ 0''.0088 \sin (\zeta - 2\tau + \xi) - 0''.0137 \sin (\zeta + 2\tau - \xi) + 0''.0022 \sin (\zeta - 2\tau - \xi)\\ &- 0''.0007 \sin (\zeta + 2\tau + \xi) + 0''.0003 \sin (\zeta - 4\tau + \xi).\end{aligned}$$

To compute the terms due to the direct action of the planets, we take for Venus,

$$\frac{m''}{m'} = \frac{1}{408134}, \qquad \gamma'' = \tan (3°23'34''), \qquad \Omega'' = 75°21',$$

for Mars,

$$\frac{m''}{m'} = \frac{1}{3200900}, \qquad \gamma'' = \tan (1°\,51'), \qquad \Omega'' = 48°\,24',$$

for Jupiter,

$$\frac{m''}{m'} = \frac{1}{1050}, \qquad \gamma'' = \tan (1°\,18'\,35''), \qquad \Omega'' = 98°\,57',$$

for Saturn,

$$\frac{m''}{m'} = \frac{1}{3512}, \qquad \gamma'' = \tan (2°\,29'), \qquad \Omega'' = 112°\,21'.$$

The quantities depending on the ratio of the mean distances are taken from Runkle's *Tables of the Coefficients of the Perturbative Function.* Thus we obtain for the several planets, in their order the values of log K expressed in seconds of arc;

$$\log K = 96.9867, \quad \log K = 95.2450n, \quad \log K = 96.1878n, \quad \log K = 95.1081n.$$

Then the action of Venus produces the following terms:

$$\begin{aligned} \delta\lambda &= -0''.0121 \sin(\zeta' - \eta) + 0''.0106 \sin(\zeta' + \eta), \\ \delta\beta &= -0''.2412 \sin \zeta' \qquad + 0''.0078 \sin(\zeta' - 2\tau). \end{aligned}$$

The action of Mars produces the terms

$$\begin{aligned} \delta\lambda &= +0''.0003 \sin(\zeta'' - \eta) - 0''.0002 \sin(\zeta'' + \eta), \\ \delta\beta &= +0''.0044 \sin \zeta''. \end{aligned}$$

The action of Jupiter produces the terms

$$\begin{aligned} \delta\lambda &= +0''.0019 \sin(\zeta''' - \eta) - 0''.0016 \sin(\zeta''' + \eta), \\ \delta\beta &= +0''.0383 \sin \zeta''' \qquad - 0''.0012 \sin(\zeta''' - 2\tau). \end{aligned}$$

The action of Saturn produces the terms

$$\begin{aligned} \delta\lambda &= +0''.0002 \sin(\zeta^{IV} - \eta) - 0''.0001 \sin(\zeta^{IV} + \eta), \\ \delta\beta &= +0''.0031 \sin \zeta^{IV}. \end{aligned}$$

The terms having the same period in the indirect and direct actions of the planets may be united in a single term, and we have

$$\begin{aligned} \zeta - \eta &= \Omega - \Pi + 90^\circ, & \zeta + \eta &= 2\,☾ - \Omega - \Pi + 90^\circ, \\ \zeta' - \eta &= \Omega - \Omega'', & \zeta' + \eta &= 2\,☾ - \Omega - \Omega''. \end{aligned}$$

Thus, preserving only the terms whose coefficients exceed $0''.01$, the value of $\delta\lambda$ due to both the indirect and direct action of the planets, is

$$\begin{aligned} \delta\lambda &= +0''.0305 \sin \Omega - 0''.2838 \cos \Omega \\ &\quad + 0''.0100 \sin(2☾ - \Omega) - 0''.0697 \cos(2☾ - \Omega) \\ &= 0''.2854 \sin(\Omega + 276^\circ 8') + 0''.0704 \sin(2☾ - \Omega + 278^\circ). \end{aligned}$$

In the case of the latitude we may write the true orbit longitude L of the moon in place of the mean, in the principal term, and neglect the remaining terms. Thus the value of $\delta\beta$, due to both actions of the planets, is

$$\begin{aligned} \delta\beta &= -0''.2256 \sin L + 1''.5802 \cos L \\ &= 1''.5963 \sin(L + 98^\circ 8'.6). \end{aligned}$$

The terms in $\delta\lambda$ and $\delta\beta$ which involve sin ☊ and sin L coalesce with the principal inequalities which are due to the figure of the earth and have the same arguments. Hansen (*Tables de la Lune*, pp. 8, 15) has, respectively, in the perturbed mean anomaly and latitude, the terms $+ 7''.760$ sin $(184°42' - ☊)$ and $+ 8''.764$ sin $(L + 169°51')$. The parts of these which depend on cos ☊ and cos L are $- 0''$ 636 cos ☊ and $+ 1''.544$ cos L. In the *Darlegung* he gives coefficients somewhat different. As to $\delta\beta$, Hansen's value nearly coincides with mine, but his coefficient in $\delta\lambda$ is more than double mine. This discrepancy is probably to be attributed to the difference of the systems of coordinates employed.*

The values of these terms which Sir G. B. Airy has determined from observation, in his first memoir on the correction of the lunar elements (*Mem. Astr. Soc.*, Vol. XVII) are

$$\delta\lambda = - 0''.97 \cos ☊, \qquad \delta\beta = + 2''.17 \cos L.$$

These he has changed to

$$\delta\lambda = - 1''.06 \cos ☊, \qquad \delta\beta = + 1''.93 \cos L,$$

in his second memoir (*Mem. Astr.* Soc., Vol. XXIX).

* It seems this suggestion is unfounded.

MEMOIR No. 40.

Elements and Perturbations of Jupiter and Saturn.

(*Astronomische Nachrichten*, Vol. CXIII, pp. 273-302, 1886.)

For several years an investigation of the motions of Jupiter and Saturn has been in progress in the Office of the American Ephemeris and Nautical Almanac, with the view of constructing tables for these two planets. The method followed is that of Hansen in his "Auseinandersetzung", except that one modification was made. In this method, as Hansen has given it, all the expressions appertaining to each planet would appear as functions of its excentric anomaly. Thus, whenever two expressions, the one belonging to Jupiter, the other to Saturn, are to be multiplied together, we should fall upon a product involving two independent variables, unless one of the factors was previously transformed so as to involve the independent variable of the other. Hence, in order to escape these troublesome and frequent transformations, the mean anomalies, or what amounts to the same thing, the time has been adopted as the independent variable.

Thus the shape, in which the final results appear, does not differ from that of Hansen's "Gegenseitige Störungen des Jupiter und Saturn", but the method of elaborating them is the more refined one of the "Auseinandersetzung".

The approximation, in this work, has been pushed to a much greater extent than in any previous treatment of the subject. And, on account of the smallness of the limit set as to terms which might be neglected, more time was consumed in computing the terms of three dimensions with respect to disturbing forces than in computing those of two dimensions.

A detailed exposition of this investigation will appear in a future volume of the Astronomical Papers of the American Ephemeris. But the formulæ for the coordinates of the two planets having now been obtained, and a preliminary comparison of them with observation made for the purpose of ascertaining what corrections the perturbations might need on account of errors in the provisionally assumed elements, the results are so satisfactory that I have thought the details of this comparison together with the final expressions for the coordinates might interest astronomers.

The elements of the two planets which were employed for the computation of the perturbations and which are to be corrected by comparison

with observation, together with the adopted values of the disturbing masses, are the following:

Epoch 1850 Jan. 0.0 Greenw. M. T.

$L = 159° \ 56' \ 26''.60$	$L' = 14° \ 49' \ 34''.04$
$\pi = 11 \ \ 56 \ \ 9.33$	$\pi' = 90 \ \ 6 \ \ 46.22$
$\Omega = 98 \ \ 56 \ \ 19.79$	$\Omega' = 112 \ \ 20 \ \ 49.05$
$i = 1 \ \ 18 \ \ 42.10$	$i' = 2 \ \ 29 \ \ 40.19$
$e = 0.04824277$	$e' = 0.05605688$
$n = 109256''.55563$	$n' = 43996''.07844$

Mercury	1 : 5000000	Jupiter	1 : 1047.879
Venus	1 : 425000	Saturn	1 : 3501.6
Earth	1 : 322800	Uranus	1 : 21000
Mars	1 : 3093500	Neptune	1 : 19700

As it was known that the adopted planes of the orbits represented the observed latitudes of the planets quite closely, comparison was made only with normals in heliocentric longitude, formed about the time of opposition. The labor of comparison without the assistance of tables is very great, and I have been obliged to be content with a very small number of normals. There are only as many as are absolutely necessary for our purpose. This is to be regretted, as if the number could have been doubled the results would have been more satisfactory.

In forming the normals Greenwich observations, taken precisely as they stand in the published volumes, without the application of any corrections, have been exclusively employed. Before 1830 the data have been derived from the Reduction of the Greenwich Observations of the Planets from 1750 to 1830. After 1830 the tabular longitude is from the English Nautical Almanac. Equal weights have been assigned to all the observations, and afterwards, in the discussion, all the normals have received equal weight.

We take up Saturn first as the discussion of this planet will give us some information as to the mass of Uranus which will be of service afterwards in treating Jupiter. The normals follow:

Greenw. M. T.	Obs.	Tab. Long.	Corr.	Hel. Long. fr. Obs.
1753 June 24.0	5	272° 54′ 10″.69	−18″.36	272° 53′ 52″.33
1757 Aug. 11.0	7	318 47 10.89	−17.82	318 46 53.07
1761 Oct. 2.5	7	8 7 58.71	+ 0.30	8 7 59.01
1811 June 15.0	5	263 22 22.66	− 6.31	263 22 16.35
1822 Oct. 30.0	6	36 40 22.56	+13.86	36 40 36.42
1837 May 4.0	10	223 50 29.0	− 1.74	223 50 27.26
1844 July 26.0	11	303 57 52.1	+11.99	303 58 4.09
1851 Oct. 24.0	12	30 49 43.9	+10.48	30 49 54.38
1858 Jan. 15.0	13	114 54 24.4	− 9.29	114 54 15.11
1866 Apr. 29.0	12	219 1 5.2	− 4.81	219 1 0.39
1874 Aug. 3.0	12	310 57 53.6	+ 8.17	310 58 1.77
1882 Nov. 15.0	9	52 42 8.9	− 7.35	52 42 1.55

Next I give some details as to the calculated longitude.

Perturbations of $n's'$ by Jupiter	Uranus	Jup.×Ur.	Neptune	Sum	$n's'$	f'	π' +prec.+nut.	Red. to Eclip.	Calculat. Long.
′ ″	″	″	″	′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	° ′ ″
−33 27.245	−56.249	+29.860	−0.730	−33 54.36	184 35 43.99	184 6 52.44	88 46 5.35	+0 59.31	272 53 57.10
−36 26.311	−42.067	+28.496	+1.189	−36 38.70	235 2 25.79	229 59 9.61	88 49 11.56	−1 19.83	318 47 1.34
−43 59.808	− 7.428	+26.439	+2.278	−43 38.51	285 33 54.17	279 16 16.34	88 52 37.27	−0 44.30	8 8 9.31
−34 54.392	−45.461	+25.517	−0.218	−35 14.55	173 2 51.58	173 46 28.19	89 34 25.98	+1 22.38	263 22 16.55
−43 10.712	−42.490	+20.603	−3.207	−43 35.81	311 55 59.59	306 55 35.95	89 44 11.85	+0 47.62	36 40 35.42
−52 49.437	− 2.190	+22.848	+2.479	−52 26.30	129 7 19.00	133 53 19.05	89 56 0.00	+1 6.96	223 50 26.01
−40 24.661	−41.397	+20.990	+0.556	−40 44.51	217 39 1.19	218 56 8.67	90 2 30.74	−0 38.74	308 58 0.67
−46 12.502	−10.227	+17.503	−0.288	−46 5.51	306 5 43.32	300 41 23.28	90 7 59.85	+0 28.48	30 49 51.61
−53 55.680	+27.624	+17.479	−3.087	−53 13.66	22 5 49.61	24 40 47.77	90 13 34.17	−0 8.48	114 54 13.46
−43 37.181	− 6.234	+19.561	+0.346	−43 23.51	123 30 35.67	128 39 33.20	90 20 28.99	+0 53.28	219 0 55.47
−22 31.188	−20.115	+17.024	+0.954	−22 33.33	224 50 18.14	220 31 40.13	90 27 14.83	−0 58.57	310 57 56.39
−33 15.246	+17.727	+13.680	+0.618	−32 43.22	325 55 4.16	322 5 59.24	90 34 30.96	+1 24.68	52 41 54.88

The equations of condition, under three different suppositions, are

$\Delta L'$	$(100\Delta n')$	$\Delta e'$	$e'\Delta\pi'$	Supp. I.		Supp. II.		Supp. III.
0.896	−0.8644	−0.140	+1.864	= − 4″.77	or	− 6″.52	or	− 6″.89
0.934	−0.8626	−1.509	+1.184	= − 8.27	"	− 9.21	"	− 9.38
1.023	−0.9026	−1.989	−0.410	= −10.30	"	− 8.86	"	− 8.67
0.896	−0.3453	+0.211	+1.857	= − 0.20	"	− 1.52	"	− 1.55
1.073	−0.2917	−1.631	−1.312	= + 1.00	"	− 0.74	"	− 0.42
0.928	−0.1175	+1.418	+1.281	= + 1.25	"	+ 2.67	"	+ 3.09
0.913	−0.0496	−1.094	+1.544	= + 3.42	"	+ 2.04	"	+ 2.23
1.063	+0.0193	−1.750	−1.125	= + 2.77	"	+ 3.34	"	+ 3.94
1.110	+0.0892	+0.859	−1.957	= + 1.65	"	+ 5.36	"	+ 6.39
0.936	+0.1528	+1.539	+1.149	= + 4.92	"	+ 5.84	"	+ 6.33
0.921	+0.2265	−1.276	+1.411	= + 5.38	"	+ 5.17	"	+ 4.93
1.095	+0.3602	−1.260	−1.705	= + 6.67	"	+ 9.22	"	+ 9.19

Supposition I is obtained by subtracting the calculated from the observed longitudes. The remaining suppositions will be explained shortly. The normal equations resulting from these equations are

$\Delta L'$	$(100\Delta n')$	$\Delta e'$	$e'\Delta\pi'$	Supp. I.		Supp. II.		Supp. III.
11.655	−2.414	−6.836	+2.350	= + 4″.27	or	+ 8″.50	or	+11″.28
−2.414	+2.739	+3.043	−3.064	= +24.58	"	+27.83	"	+28.07
−6.836	+3.043	+21.554	+3.830	= +18.87	"	+24.41	"	+25.44
+2.350	−3.064	+3.830	+25.555	= −13.73	"	−30.67	"	−33.43

The solution of these equations gives

	I.		II.		III.
$\Delta L' =$	+ 2″.692	or	+ 3″.692	or	+ 4″.087
$\Delta n' =$	+ 0.12285	"	+ 0.12727	"	+ 0.12750
$\Delta e' =$	− 0″.131	or	+ 0″.523	or	+ 0″.723
$e'\Delta\pi' =$	+ 0.708	"	− 0.093	"	− 0.265

The residuals (Obs.—Calc.), severally in the three suppositions, are

	I.	II.	III.
1753 June 24.0	+2″.10	+1″.41	+1″.06
1757 Aug. 11.0	−1.22	−0.78	−0.79
1761 Oct. 2.5	−1.93	−0.15	−0.01
1811 June 15.0	+0.35	−0.36	−0.45
1822 Oct. 30.0	+2.41	−0.25	−0.25
1837 May 4.0	−0.53	+0.11	+0.10
1844 July 26.0	+0.34	+0.02	+0.34
1851 Oct. 24.0	+0.24	−0.02	+0.31
1858 Jan. 15.0	−0.94	−0.51	−0.43
1866 Apr. 29.0	−0.09	−0.26	−0.27
1874 Aug. 3.0	−1.05	−0.32	−0.44
1882 Nov. 15.0	+0.35	+1.09	+0.59

The residuals of Supposition I are not altogether satisfactory, and on comparing them with the portions of the perturbations which are proportional to the mass of Uranus it is suggested that a better agreement would be obtained by diminishing this mass. Hence I concluded to put the value at 1:22640, which is about the average of all the results which have been obtained from the observations of the satellites at the Washington Observatory. This has given rise to the numbers of the column headed Supposition II. It will be seen that the residuals of II are fairly satisfactory, and it does not seem worth while in this preliminary investigation to inquire whether we should do better with another value of the mass of Uranus.

The perturbations being now corrected for the changes in the elements shown by II and for the similar ones to be given hereafter for Jupiter, the resulting numbers appear under Supposition III, to which we hold as being the best which can be done at present. The residuals of III are, to some extent, better than those of II.

We pass now to Jupiter. The normals are formed as follows:

Greenw. M. T.	Obs.	Tab. Long.	Corr.	Hel. Long. fr. Obs.
1757 May 3.5	7	223° 44′ 36″.85	+ 6″.59	223° 44′ 43″.44
1759 July 9.5	8	287 33 42.20	+10.70	287 33 52.90
1819 Aug. 5.5	12	312 16 54.91	+ 6.78	312 17 1.69
1855 Aug. 22.0	16	327 44 57.70	− 5.46	327 44 52.24
1858 Dec. 16.0	9	77 11 8.30	+ 5.87	77 11 14.17
1861 Feb. 16.0	11	142 29 48.10	+ 8.31	142 29 56.41
1864 May 16.9	9	232 58 30.70	+17.35	232 58 48.05
1867 Aug. 23.0	6	332 18 32.80	+ 0.77	332 18 33.57
1870 Dec. 19.0	6	81 53 54.70	+ 7.63	81 54 2.33
1874 Mar. 18.0	12	176 56 16.60	+ 7.27	176 56 23.87
1877 June 19.0	11	268 41 48.00	+15.26	268 42 3.26
1878 July 20.0	7	301 49 21.10	− 0.17	301 49 20.93
1880 Oct. 7.0	12	14 30 48.20	+ 0.18	14 30 48.38

In getting the calculated longitude the mass of Uranus has been made 1:22640. The details are as follows:

Perturbations of ns by Saturn	Uranus	Sat.×Ur.	Neptune	Sum	ns	f	ϖ +prec.+nut.	Red. to Eclip.	Calculat. Long.
′ ″	″	″	″	′ ″	° ′ ″	° ′ ″	° ′ ″	″	° ′ ″
+18 36.578	−0.140	−8.244	−0.389	+18 27.80	216 13 1.63	213 6 14.83	10 38 20.17	+25.85	223 45 0.85
+14 0.876	+0.206	−8.205	−0.104	+13 52.77	282 21 51.78	276 54 7.47	10 40 6.13	− 8.92	287 34 4.68
+12 31.844	−0.164	−6.551	−0.061	+12 25.07	305 26 34.70	300 46 53.00	11 30 38.19	−25.01	312 17 6.18
+19 50.624	+0.518	−5.380	+0.057	+19 45.82	319 30 7.07	315 44 32.43	12 0 43.60	−26.78	327 44 49.25
+18 3.316	+0.047	−4.846	−0.060	+17 58.46	60 10 43.08	65 7 0.23	12 3 47.01	+18.66	77 11 5.90
+19 59.079	−0.164	−5.161	−0.184	+19 53.57	126 6 6.76	130 24 39.61	12 5 45.95	−26.96	142 29 58.60
+19 8.804	+0.052	−5.609	+0.172	+19 3.42	224 38 3.46	220 50 7.14	12 8 23.81	+26.97	232 58 57.92
+ 8 15.384	+1.595	−4.957	−0.029	+ 8 11.99	323 34 51.04	320 8 1.98	12 10 52.52	−25.89	332 18 28.61
+14 13.009	−0.098	−4.496	−0.004	+14 8.41	64 33 9.08	69 40 12.63	12 13 26.05	+15.28	81 53 53.96
+22 54.849	−0.133	−5.074	−0.091	+22 49.55	168 9 37.02	164 40 26.55	12 16 16.14	−11.17	176 56 31.52
+13 19.365	−1.185	−5.089	+0.239	+13 13.33	261 47 44.17	256 22 43.46	12 19 17.02	+ 9.68	268 42 10.16
+ 9 33.806	−0.584	−4.830	+0.148	+ 9 28.54	294 38 14.13	289 29 23.34	12 20 17.59	−19.13	301 49 21.80
+ 9 27.323	+1.446	−4.256	−0.094	+ 9 24.42	1 56 23.81	2 8 20.61	12 22 12.40	+ 4.92	14 30 37.93

The equations of condition, under three different suppositions, are:

ΔL	$(100\Delta n)$	Δe	$e\Delta\pi$		Supp. I.		Supp. II.		Supp. III.
$0.924\,\Delta L$	$-0.8562\,(100\Delta n)$	$-1.073\,\Delta e$	$+1.575\,e\Delta\pi$	$=$	$-17.''41$	or	$-23.''12$	or	$-17.''41$
1.015	−0.9184	−1.996	−0.315	=	−11.78	"	−16.49	"	−11.76
1.054	−0.3204	−1.744	−1.113	=	− 4.49	"	− 8.86	"	− 4.55
1.074	+0.0606	−1.422	−1.537	=	+ 2.99	"	− 4.11	"	+ 2.97
1.045	+0.0936	+1.837	−0.926	=	+ 8.27	"	+ 1.99	"	+ 8.14
0.942	+0.1048	+1.502	+1.209	=	− 2.19	"	− 8.45	"	− 2.37
0.932	+0.1339	−1.286	+1.419	=	− 9.87	"	−15.80	"	− 9.97
1.079	+0.1904	−1.309	−1.641	=	+ 4.96	"	+ 2.01	"	+ 4.96
1.038	+0.2175	+1.896	−0.776	=	+ 8.37	"	+ 3.47	"	+ 8.27
0.912	+0.2208	+0.517	+1.818	=	− 7.65	"	−14.61	"	− 7.80
0.981	+0.2694	−1.936	+0.397	=	− 6.90	"	−11.24	"	− 6.93
1.036	+0.2958	−1.905	−0.747	=	− 0.87	"	− 4.15	"	− 0.85
1.103	+0.3392	+0.077	−2.125	=	+10.45	"	+ 6.99	"	+10.47

The normal equations, resulting from these equations, are

ΔL	$(100\Delta n)$	Δe	$e\Delta\pi$		Supp. I.		Supp. II.		Supp. III.
$13.318\,\Delta L$	$-0.097\,(100\Delta n)$	$-7.008\,\Delta e$	$-3.836\,e\Delta\pi$	$=$	$-21.''28$	or	$-87.''62$	or	$-21.''97$
−0.097	+2.128	+2.599	−1.481	=	+29.07	"	+30.74	"	+28.97
−7.008	+2.599	+30.443	+3.462	=	+91.62	"	+116.90	"	+91.12
−3.836	−1.481	+3.462	+22.344	=	−100.44	"	−98.30	"	−100.84

And their solution gives

	I.		II.		III.
$\Delta L =$	$-1.''540$	or	$-6.''923$	or	$-1.''615$
$\Delta n =$	$+0.07188$	"	$+0.07655$	"	$+0.07153$
$\Delta e =$	$+2.574$	"	$+2.210$	"	$+2.546$
$e\Delta\pi =$	-4.683	"	-5.424	"	-4.711

The residuals (Obs.—Cal.), severally in the three suppositions, are

	I.	II.	III.
1757 May 3.5	+0".30	+0".75	+0".35
1759 July 9.5	+0.05	+0.27	+0.05
1819 Aug. 5.5	−1.29	−1.31	−1.36
1855 Aug. 22.0	+0.66	−2.33	+0.65
1858 Dec. 16.0	+0.15	−0.58	+0.12
1861 Feb. 16.0	+0.31	+0.51	+0.28
1864 May 16.0	+0.55	+0.16	+0.53
1867 Aug. 23.0	+0.93	+2.01	+0.94
1870 Dec. 19.0	−0.10	+0.58	−0.10
1874 Mar. 18.0	−0.66	−1.26	−0.67
1877 June 19.0	−0.49	−0.08	−0.48
1878 July 20.0	0.00	+0.92	+0.05
1880 Oct. 7.0	−0.44	+0.33	−0.39

Supposition I corresponds to Bessel's value 1:3501.6 of the mass of Saturn, while II results from using the value 1:3482.2 recently derived by Prof. A. Hall from observations of Japetus. The residuals of II are generally larger than those of I, and, in consequence, I shall hold to Bessel's value, although it is possible that when the observations are more properly reduced a better showing may result for the larger mass. In fine Supposition III results from I by applying to the perturbations the corrections due to the adopted changes in the elements.

Thus we have, as the result of this investigation, the following elements of Jupiter and Saturn suited to Hansen's form for the perturbations:

Epoch 1850 Jan. 0.0 Greenw. M. T.

$L = 159°\ 56'\ 24''.98$	$L' = 14°\ 49'\ 38''.13$
$\pi = 11\ \ 54\ \ 31.67$	$\pi' = 90\ \ 6\ \ 41.50$
$\Omega = 98\ \ 56\ \ 17.79$	$\Omega' = 112\ \ 20\ \ 49.05$
$i = 1\ \ 18\ \ 42.10$	$i' = 2\ \ 29\ \ 40.19$
$e = 0.04825511$	$e' = 0.05606038$
$n = 109256''.62716$	$n' = 43996''.20594$
$\log a = 0.7162374048$	$\log a' = 0.9794956985$

As it may be desired to compare these elements with other determinations derived on the supposition that the perturbations are to be added directly to the true longitude, it may be well to note that before this comparison is made, certain corrections need to be applied to them. To derive these we compute some of the terms of the expression

$$\delta f = \frac{df}{dg}\, n\delta z + \tfrac{1}{2}\frac{d^2 f}{dg^2}(n\delta z)^2.$$

For Jupiter it will be sufficient to take

$$n\delta z = -0''.193 \sin 2g + 0''.136 \cos 2g - 0''.74152t \cos g - 0''.00890t \cos 2g,$$
$$(n\delta z)^2 = +3''.761 - 0''.205 \cos g + 0''.824 \sin g,$$

and for Saturn

$$n'\delta z' = -1''.361 \sin 2g' + 2''.229 \cos 2g' - 0''.019 \sin 3g' + 0''.648 \cos 3g'$$
$$- 2''.2821t \cos g' - 0''.0317t \cos 2g',$$
$$(n'\delta z')^2 = +22''.30 + 8''.356 \cos g' + 4''.894 \sin g' - 0''.187 \cos 2g' - 0''.462 \sin 2g'$$
$$+ 0''.010 \cos 3g' - 0''.460 \sin 3g' + 0''.00120t \sin g'.$$

With these values it is found that δf and $\delta f'$ contain severally the terms,

$$\delta f = -0''.020 - 0''.03580t - 0''.189 \sin g + 0''.005 \cos g$$
$$\delta f' = -0''.125 - 0''.12804t - 1''.364 \sin g' + 0''.121 \cos g'.$$

As in the second method of perturbations these terms would be included in the elliptic portions of the coordinates, we must apply to the preceding values of the elements the corrections

$$\Delta L = -0''.02 \qquad \Delta L' = -0''.125$$
$$\Delta \pi = -0''.05 \qquad \Delta \pi' = -1''.08$$
$$\Delta e = -0.00000046 \qquad \Delta e' = -0.00000331$$
$$\Delta n = -0''.03580 \qquad \Delta n' = -0.''12804.$$

Then the elements, changed to suit the second form of the perturbations, are

$$L = 159° \, 56' \, 24''.96 \qquad L' = 14° \, 49' \, 38''.00$$
$$\pi = 11 \;\; 54 \;\; 31.62 \qquad \pi' = 90 \;\; 6 \;\; 40.42$$
$$e = 0.04825465 \qquad e' = 0.05605707$$
$$n = 109256''.59136 \qquad n' = 43996''.07790$$
$$\log a = 0.7162374992 \qquad \log a' = 0.9794965411$$

We now proceed to explain the formulæ for the heliocentric coordinates of Jupiter and Saturn. As the mass of Uranus has been modified, it seemed well to make some further changes. Thus we have put

Mercury 1:7500000, Venus 1:408134, Earth 1:327000.

These give for the motion of the plane of the ecliptic the formulæ

$$\sin i_0 \sin \Omega_0 = +\;\; 5''.2723T + 0''.19501T^2 - 0''.000240T^3$$
$$\sin i_0 \cos \Omega_0 = -46.7608T + 0.05666T^2 + 0.000506T^3$$

where the unit of T is a century of Julian years and it is counted from 1850.0. The value of the general precession employed is

$$\psi' = 5025''.7870T + 1''.10739T^2 + 0''.000174T^3 - 0''.0000488T^4 - 0''.00000023T^5.$$

The values of the constituents of the arguments, occurring in the formulæ, are

$$
\begin{array}{rll}
g & = 148^\circ\,1'\,50''.60 + 109256''.62716t & \\
g' & = 284\;42\;56.63 + 43996.20594t & \\
g'' & = 220\;10\;10.35 + 15425.752\;\;t & \text{(Newcomb, Orbit of Uranus, p. 181)} \\
g''' & = 291\;48\;8.61 + 7864.935\;\;t & \text{(Newcomb, Orbit of Neptune, p. 76)} \\
\text{Venus} - \text{Jupiter} & = 84^\circ\,1' + 1997384''.73\;t & \\
\text{Earth} - \text{Jupiter} & = 299\;52 + 1186720.79\;t & \\
\text{Venus} - \text{Saturn} & = 229\;\;8 + 2062645.15\;t & \\
\text{Earth} - \text{Saturn} & = 84\;59 + 1251981.21\;t &
\end{array}
$$

It will be perceived that the value of g does not agree with that derived from the elements previously given. This results from the fact that the value $\pi = 11^\circ\,54'\,34''.38$ was used in getting the quantities K. Hence in order to employ g as derived from the given elements, it would be necessary to correct K by $-\,2''.71\,i$, if the argument contains ig. To avoid this, for the perturbations, we simply count g from the old place of the perihelion.

The values of $n\delta z$ and $\Delta\beta$ are given the form

$$k_0 \sin(\chi + K_0) + k_1 T \sin(\chi + K_1) + k_2 T^2 \sin(\chi + K_2) + k_3 T^3 \sin(\chi + K_3),$$

and that of com. log $\left(\frac{\bar{r}}{r} = 1 + \nu\right)$, the form

$$k_0 \cos(\chi + K_0) + k_1 T \cos(\chi + K_1) + k_2 T^2 \cos(\chi + K_2) + k_3 T \;\cos(\chi + K_3).$$

K is so taken that k may be positive, except in the absolute terms, where K is supposed to vanish and k receives its proper sign. It will be noticed that, in some places, the arguments $5g' - 2g$ and $10g' - 4g$ have their motions equated. A greater degree of exactitude is thus obtained without augmenting the usual number of terms. The t, in these places, must be counted from the epoch of the elements.

The formula, for the latitude referred to the ecliptic of date, is $\beta = \beta_0 + \Delta\beta$; and l denotes the orbit longitude $= f + \pi$. It will be noticed that the reduction to the ecliptic has no terms involving both g and g'. This is because all these terms, after having been multiplied by $\frac{\bar{r}^2}{a^2\sqrt{1 - e^2}}$, have been added to $n\delta z$. And care has been taken to rectify log $\frac{\bar{r}}{r}$ and $\Delta\beta$ on this account.

Perturbations of Jupiter: $n\delta z$.

χ	k_0	K_0	k_1	K_1	k_2	K_2	k_3	K_3
g' g					—0.27766		+0″016021	
—1			100″6354	227° 27′ 47″17	60266	302° 34′.5	364	47° 41′
—2	0″236	35° 8′	1.2132	227 10.6	2171	284 38	4	45
—3	0.047	137	312	228 2	74	281		
—4	0.002	103	9	227				
1 + 3	0.005	147						
1 + 2	0.128	123 20	57	21 16				
1 + 1	1.237	215 14.1	332	115 58				
1	11.156	150 56 7″	1755	49 46	68	321 43		
1 — 1	79.843	79 12 4	45	244 58				
1 — 2	1.508	90 37.5	237	131 4				
1 — 3	0.108	108 27	26	197 47				
1 — 4	0.018	212 27						
2 + 2	0.013	205 33	7	123				
2 + 1	0.487	184 19	211	86 4				
2	6.813	123 49.3	1753	13 43	42	228		
2 — 1	123.012	1 24 42.0	1.2671	301 24.2	700	216 42		
2 — 2	194.634	336 53 36.8	222	354 34	17	31		
2 — 3	2.811	331 31.5	652	22 47				
2 — 4	0.054	305 46	28	13 20				
2 — 5	0.002	300						
3 + 1	0.062	275 52	29	185 11				
3	3.685	270 58.7	1418	174 10				
3 — 1	14.038	312 11 28	2316	210 12.5	171	161 53		
3 — 2	82.649	127 22 45	1.1498	30 0.9	609	299 34		
3 — 3	16.228	57 42 35	147	150 34	6	297		
3 — 4	0.405	38 13	78	101 47				
3 — 5	0.014	327 36	4	50				
4	0.015	177 16						
4 — 1	0.684	191 30	304	84 0				
4 — 2	16.838	98 27 55	4607	0 32.8	313	260 45		
4 — 3	14.978	26 2 27	2044	288 17.1	121	197 39		
4 — 4	3.611	129 27.3	39	36 49				
4 — 5	0.152	104 21	24	168 36				
4 — 6	0.009	33						
5	0.004	45	73	17 23				
5 — 1	0.776	1 46.6	2566	11 51.6	1295	283 55		
$\left\{ \begin{matrix} 5-2 \\ -81''97009t \end{matrix} \right\}$	1196.138	67 9 4.42	5.5814	247 9.1	15560	48 49		
5 — 3	160.938	176 27 37.4	4.7607	80 53.5	5921	349 22		
5 — 4	3.666	133 33.3	810	75 27	89	108 25		
5 — 5	1.121	206 52.0	16	144 22				
5 — 6	0.068	178 43	9	245				
5 — 7	0.004	120						
6 — 1	0.004	320						
6 — 2	0.150	29 31	88	290 27				
6 — 3	1.181	150 52.7	944	289 28	12	315		
6 — 4	1.522	74 35.7	398	336 28				
6 — 5	0.803	179 12	114	82 54				
6 — 6	0.373	285 43	3	158				
6 — 7	0.032	254 31	4	310				
6 — 8	0.002	225						
7 — 2	0.008	213	15	88 4				
7 — 3	1.916	214 9.7	775	116 9.9	31	0		
7 — 4	2.897	223 47.4	1111	125 23.6	46	212 21		

χ	k_0	K_0	k_1	K_1
g' g				
7 — 5	0″.294	161° 33′	0″.0093	64° 34′
7 — 6	0.305	258 47	41	159 35
7 — 7	0.138	2 15	1	270
7 — 8	0.015	329 45	2	342
7 — 9	0.001	301		
8 — 2	0.010	340 29		
8 — 3	0.278	198 1	132	104 13
8 — 4	1.862	13 32.4	878	277 18
8 — 5	0.319	304 24	132	207 55
8 — 6	0.137	234 50	44	139 1
8 — 7	0.124	336 32	14	238 50
8 — 8	0.054	77 42		
8 — 9	0.008	47		
8 — 10	0.001	16		
9 — 3	0.009	170		
9 — 4	0.528	344 38	281	247 56
9 — 5	0.504	272 23	251	175 17
9 — 6	0.107	14 51	35	280 37
9 — 7	0.063	312 29	17	218 50
9 — 8	0.054	53 34	7	318
9 — 9	0.022	154 15		
9 — 10	0.004	124		
{ 10 — 4 / —145″.72t }	11.024	313 40.9	876	133 41
		k_2 = 0.01338	K_2 = 311° 27′	
10 — 5	3.578	63 17.8	2075	325 49.8
10 — 6	0.097	16 23	44	289 54
10 — 7	0.034	93 31	11	352
10 — 8	0.030	28 18	8	285
10 — 9	0.025	129 28		
10 — 10	0.009	230		
10 — 11	0.002	201		
11 — 4	0.005	286		
11 — 5	0.097	34 14	29	294 49
11 — 6	0.079	321 52	29	225 9
11 — 7	0.040	66 1	10	328
11 — 8	0.012	168 13	1	90
11 — 9	0.015	104 10	3	0
11 — 10	0.012	208 35		
11 — 11	0.004	304		
11 — 12	0.001	276		
12 — 5	0.065	35 13	28	266 49
12 — 6	0.055	293 31	30	190 14
12 — 7	0.023	38 45	4	293
12 — 8	0.017	144 9	4	40
12 — 9	0.004	223	2	198
12 — 10	0.007	184		

χ	k_0	K_0	k_1	K_1
g' g				
12 — 11	0″.005	284°		
12 — 12	0.002	12		
g'' g				
1 + 1	0.010	183		
1	0.273	174 41′		
1 — 1	0.910	156 57		
1 — 2	0.006	188		
2	0.010	190		
2 — 1	0.519	136 42		
2 — 2	0.464	132 49		
2 — 3	0.012	130 44		
3	0.001	235		
3 — 1	0.091	132 12		
3 — 2	0.145	126 54		
3 — 3	0.034	287 32		
3 — 4	0.002	283		
4 — 1	0.015	128 38		
4 — 2	0.034	121 9		
4 — 3	0.013	282 16		
4 — 4	0.004	83		
5 — 1	0.003	127		
5 — 2	0.008	115		
5 — 3	0.003	277		
5 — 4	0.002	78		
5 — 5	0.001	237		
6 — 1	0.001	117		
6 — 2	0.002	109		
6 — 3	0.001	270		
6 — 4	0.001	72		
7 — 1	0.015	116 6		
7 — 2	0.004	103		
g' g g''				
6 — 3 — 3	0.472	105 59	0″.0072	337° 27′
6 — 2 — 3	8.749	187 49.9	2864	64 10
g''' g				
1	0.011	99 21		
1 — 1	0.286	31 37		
1 — 2	0.004	35		
2	0.002	61		
2 — 1	0.178	243 29		
2 — 2	0.101	242 47		
2 — 3	0.002	242		
3 — 1	0.002	209		
3 — 2	0.002	151		
3 — 3	0.006	273		
♀ — ♃	0.070	0		
♂ — ♃	0.121	0		

Perturbations of Jupiter: Common log $\frac{r}{r}$. (In units of the 7[th] decimal.)

χ	k_0	K_0	k_1	K_1				
g	− 40.83		− 17.298		k_2 = − 0.024			
− 1	18.17	323° 32′	1059.426	227° 27′ 21″.3	k_2 = 6.342	K_2 = 302° 38′.2	k_3 = 0.0038	K_3 = 47° 42′
− 2	3.89	31 43	25.539	227 13.7	k_2 = 840	K_2 = 287 0	k_3 = 1	K_3 = 45

χ	k_0	K_0	k_1	K_1
g' g				
−3	0.80	133° 10′	0.958	228° 0′
		$k_2 = 0.017$	$K_2 = 282°$	
−4	0.07	111	39	227
		$k_2 = 0.001$	$K_2 = 270°$	
1 + 3	0.13	323 49		
1 + 2	2.08	308 0	81	208 37
1 + 1	16.58	33 51	451	294 30
1	46.87	341 13.9	857	229 1
		$k_2 = 0.003$	$K_2 = 149°$	
1 − 1	545.14	79 11 20″	51	236 41
1 − 2	23.70	87 58.8	289	130 59
1 − 3	2.09	107 4	55	196 40
1 − 4	0.33	206 40		
2 + 2	0.31	18 52	9	299
2 + 1	7.42	1 54	298	265 2
2	61.05	305 11.4	1.601	193 19
		$k_2 = 0.001$	$K_2 = 297°$	
2 − 1	383.02	356 11 14	2.917	300 58.4
		$k_2 = 0.021$	$K_2 = 217°$	
2 − 2	2303.37	336 53 50.7	242	352 6
		$k_2 = 0.002$	$K_2 = 135°$	
2 − 3	62.33	333 10.4	874	22 59
2 − 4	1.94	319 56	41	3
2 − 5	0.10	329		
3 + 1	1.39	94 40	58	355 38
3	43.89	90 51	1.688	353 42
3 − 1	56.45	133 2.3	858	29 1
		$k_2 = 0.001$	$K_2 = 333°$	
3 − 2	738.42	126 35 26	10.215	30 3.5
		$k_2 = 0.051$	$K_2 = 298° 56′$	
3 − 3	241.37	58 30 37	154	121 7
3 − 4	9.52	44 11	121	98 36
3 − 5	0.34	356 55	9	45
4	0.23	355 51	6	248
4 − 1	4.61	24 58	83	91 34
4 − 2	85.28	94 3.3	2.283	358 30.5
		$k_2 = 0.009$	$K_2 = 270°$	
4 − 3	193.21	27 0.4	2.652	288 26.0
		$k_2 = 0.012$	$K_2 = 197°$	
4 − 4	59.81	127 50.7	51	358 51
4 − 5	3.50	109 14	40	168 36
4 − 6	0.20	52 55		
5	0.12	215	152	197 54
5 − 1	8.14	180 47	2.691	192 9
5 − 2	229.34	237 53.5	9.058	143 57.0
		$k_2 = 0.162$	$K_2 = 46° 23′$	
5 − 3	1679.20	176 23 36	49.701	80 52.4
		$k_2 = 0.528$	$K_2 = 343° 42′$	
5 − 4	65.06	141 13.2	931	73 6
		$k_2 = 0.011$	$K_2 = 326°$	
5 − 5	20.58	204 48	42	243 34
5 − 6	1.56	184 1	17	241
5 − 7	0.11	129 51	3	207
6 − 1	0.05	137		
6 − 2	0.92	203 41	40	102 57
6 − 3	8.78	145 29	365	46 48

χ	k_0	K_0	k_1	K_1
g' g				
6 − 4	20.79	76° 42′	0.565	337° 5′
6 − 5	13.52	180 37	192	80 46
6 − 6	6.92	283 56	8	117
6 − 7	0.71	260 4	6	307
6 − 8	0.06	236		
7 − 2	0.18	7 25	19	283
7 − 3	5.50	214 14	216	118 29
7 − 4	34.30	223 11.4	1.313	125 12
7 − 5	5.17	167 54	159	68 46
7 − 6	5.43	259 28	74	158 59
7 − 7	2.68	0 22	4	147
7 − 8	0.34	335 13	3	27
7 − 9	0.03	312		
8 − 3	1.09	13 26	24	259
8 − 4	16.42	12 48	775	276 18
8 − 5	4.89	304 0	193	208 45
8 − 6	2.42	239 46	73	142 35
8 − 7	2.31	337 34	29	232 53
8 − 8	1.08	75 50	3	243
8 − 9	0.18	50 5		
9 − 3	0.08	359	3	117
9 − 4	2.61	340 31	109	240 8
9 − 5	6.53	272 59	312	175 24
9 − 6	1.75	10 57	66	275 1
9 − 7	1.18	316 50	33	211
9 − 8	1.04	54 49	16	315
9 − 9	0.45	151 37		
9 − 10	0.09	125		
10 − 4	3.47	123 36	190	31 11
10 − 5	37.04	63 10.9	2.298	325 45
10 − 6	1.81	22 44	82	296 1
10 − 7	0.68	88 13	28	356
10 − 8	0.57	33 57	15	287
10 − 9	0.49	131 12	7	31
10 − 10	0.19	226 9		
10 − 11	0.04	203		
11 − 5	0.65	31 58	17	290
11 − 6	1.10	322 57	45	220
11 − 7	0.70	66 59	31	330
11 − 8	0.25	162 23	11	79
11 − 9	0.29	112 52	9	7
11 − 10	0.23	208 41		
11 − 11	0.08	299 36		
12 − 6	0.49	296 9	37	189
12 − 7	0.39	39 39	9	299
12 − 8	0.26	145 5	4	237
12 − 9	0.09	236 55	5	346
12 − 10	0.15	186 6	3	90
12 − 11	0.11	284		
12 − 12	0.04	10		
g'' g				
1 + 1	0.12	3		
1	0.24	8		
1 − 1	8.46	156 57		
1 − 2	0.18	177		
2	0.06	114		

χ	k_0	K_0
g'' g		
2 — 1	4.55	136° 22′
2 — 2	6.70	132 49
2 — 3	0.27	130
2 — 4	0.01	132
3 — 1	0.71	131 32
3 — 2	1.96	127 7
3 — 3	0.56	287
3 — 4	0.04	285
4 — 1	0.09	125
4 — 2	0.44	122
4 — 3	0.21	282
4 — 4	0.08	83
5 — 2	0.09	116

χ	k_0	K_0	k_1	K_1
g'' g				
5 — 3	0.05	277°		
5 — 4	0.04	80		
5 — 5	0.01	239		
6 — 2	0.03	110		
6 — 3	0.01	270		
6 — 4	0.01	75		
7 — 2	0.04	103		
g' g g''				
6 — 3 — 3	4.97	105 59′	0.076	337° 27′
6 — 2 — 3	1.08	175 11		

χ	k_0	K_0
g''' g		
1	0.06	22°
1 — 1	2.83	31 37′
1 — 2	0.07	34
2	0.04	242
2 — 1	1.75	243 22
2 — 2	1.52	242 44
2 — 3	0.06	242
3 — 1	0.02	207
3 — 2	0.03	161
3 — 3	0.10	274
♀ — ♃	1.48	0
♁ — ♃	2.55	0

Perturbations of Jupiter: $\Delta\beta$.

χ	k_0	K_0	k_1	K_1
g' g				
	+0″.037			
— 2	0.015	66°		
— 3	0.001	82		
1 + 2	0.005	353		
1 + 1	0.104	8 51′	0″.0005	158°
1	0.536	325 28	70	54 16′
1 — 1	0.126	208 0	27	188 26
1 — 2	0.265	193 10	43	103 27
1 — 3	0.012	204	4	90
2 + 1	0.018	283	4	14
2	0.342	265 52	21	313
2 — 1	0.627	43 9	81	137 30
2 — 2	0.221	114 42	59	82 11
2 — 3	0.056	267	4	57
2 — 4	0.003	282	2	0
3 + 1	0.003	33	1	225
3	0.056	49	2	153
3 — 1	0.165	356 6	6	38
3 — 2	1.018	122 15	120	212 25
3 — 3	0.057	163 7	6	218
3 — 4	0.019	351	2	153
3 — 5	0.001	355		
4	0.006	22		
4 — 1	0.047	329 38		
4 — 2	0.144	99 51	7	188
4 — 3	0.247	22 4	37	109
4 — 4	0.021	342	2	90
4 — 5	0.009	60	1	135
5	0.009	111	1	315
5 — 1	0.184	111 34	36	8
5 — 2	0.194	359 38	6	288
5 — 3	3.548	174 54.4	77	327 12

χ	k_0	K_0	k_1	K_1
g' g				
5 — 4	0″.187	161° 37′	0″.0009	238°
5 — 5	0.008	125	4	104
5 — 6	0.003	136		
6 — 1	0.001	74		
6 — 2	0.007	16		
6 — 3	0.037	150		
6 — 4	0.048	74		
6 — 5	0.012	165		
6 — 6	0.003	121		
6 — 7	0.001	216		
7 — 2	0.004	337		
7 — 3	0.005	144		
7 — 4	0.053	44		
7 — 5	0.011	135		
7 — 6	0.004	245		
7 — 7	0.002	198		
7 — 8	0.001	292		
8 — 3	0.001	48		
8 — 4	0.009	201		
8 — 5	0.008	127		
8 — 6	0.004	222		
8 — 7	0.001	318		
8 — 8	0.001	90		
9 — 5	0.004	89		
9 — 6	0.003	196		
9 — 7	0.002	298		
10 — 4	0.003	66		
10 — 5	0.073	60 20		
10 — 6	0.003	106		
10 — 7	0.001	281		
10 — 8	0.001	23		

$$
\begin{aligned}
\sin\beta_0 = \sin i \sin(l - \Omega) & \\
& + 36''.7739\,T \sin(l + 23^\circ\, 33'\, 44''.2) \\
& + 0''.16385\,T^2 \sin(l + 138^\circ\, 32'.7) \\
& + 0''.000513\,T^3 \sin(l + 249^\circ\, 14').
\end{aligned}
$$

Reduction of orbit longitudes to the mean equinox and ecliptic of date

$$
\begin{aligned}
&= +\,27''.029 \sin (2l + 342^\circ 7' 20'') + 0''.002 \sin (4l + 324^\circ) \\
&\quad + [5026''.3064 + 0''.4211 \sin (2l + 104^\circ 37'.9)]\, T \\
&\quad + [1''.10640 + 0''.00351 \sin (2l + 223^\circ 9')]\, T^2 \\
&\quad + [0''.000169 + 0''.000020 \sin (2l + 340^\circ)]\, T^3 \\
&\quad - 0''.0000488\, T^4 - 0''.00000023\, T^5.
\end{aligned}
$$

Perturbations of Saturn: $n'\delta z'$.

χ	k_0	K_0	k_1	K_1	k_2	K_2	k_3	K_3
g' g								
					+0″.68075		−0″.028403	
1			269″.1355	237° 59′ 16″.93	1.79908	139° 10′.7	1820	351° 12′
2	2″.612	121° 24′.3	3.7149	238 37 4.7	12134	123 18	1214	20 4
3	0.648	91 39	972	252 37	527	119 4	88	6
4	0.026	4	54	245 39	24	118		
5	0.003	214	2	241				
−4−1	0.006	21	1	59				
−3−1	0.006	76	1	202				
−2−1	0.195	165 51	78	264 34				
−1−1	0.362	141 48	176	227 53	10	294		
−1	12.089	86 45 50″	1466	207 47	73	313		
1−1	7.196	189 34 58	2964	303 40	110	296 9		
2−1	421.948	181 25 39.47	4.1702	122 26 54	2192	38 34		
3−1	33.511	121 13 43.1	8286	31 8.2	1088	350 11		
4−1	0.101	90 31	295	12 11	103	306 56		
5−1	0.043	159 30	31	28 0	8	315		
6−1	0.003	124	1	135				
7−1	0.003	257						
−2−2	0.004	141	3	241				
−1−2	0.076	244 22	31	342				
−2	0.164	114 12	20	276	3	270		
1−2	2.764	250 7.5	387	289 13	4	122		
2−2	32.025	156 58 4	94	346 20	14	220		
3−2	26.138	135 32 59	8874	42 50.1	1185	300 40		
4−2	683.664	277 23 39.14	16.5261	179 34 44	15267	84 50.7		
{5−2 −82″.00170t}	2907.855	247 6 38.15	13.9914	67 6 38	29847	221 44.3		
6−2	1.719	255 17.2	2.0610	125 55.0	8930	27 31.4		
7−2	0.034	323 7	555	126 55	368	18 30		
8−2	0.006	339	20	128				
−1−3	0.003	208	4	289				
−3	0.029	335	10	62				
1−3	0.139	269 30	15	348				
2−3	0.190	142 54	19	345	2	0		
3−3	6.513	234 22.7	22	357	8	246		
4−3	4.600	203 15.5	660	107 20	33	11		
5−3	3.250	174 37.2	903	77 49	112	340 41		
6−3	3.339	157 20.6	1382	58 30	359	314 36		
7−3	6.247	31 24.1	2540	289 53	179	116 10		
8−3	0.654	18 10	451	303 37	34	106		
9−3	0.057	110 32	2	130				
10−3	0.002	59						
−4	0.001	291						
1−4	0.011	22	6	135				
2−4	0.021	25	5	93				

χ	k_0	K_0	k_1	K_1	k_2	K_2
g' g						
3 — 4	0″.122	205° 21′	0″.0006	356°		
4 — 4	1.910	312 8.2	4	62	0″.00004	109°
5 — 4	1.290	281 50.2	194	185 6′	11	115
6 — 4	0.692	249 33	201	152 59	17	75
7 — 4	0.375	41 51	134	300 15	16	30
8 — 4	1.486	14 35.6	774	277 44	31	203
9 — 4	8.824	163 42 12	5281	67 33.4	1228	331 39′
{10 — 4 — 148″.145t}	26.795	133 36 50.8	2274	313 36.8	5217	122 44
11 — 4	0.002	197	199	13 56	40	275

χ	k_0	K_0	k_1	K_1
g' g				
1 — 5	0″.001	0°		
2 — 5	0.006	115	0″.0002	219°
3 — 5	0.010	106	2	194
4 — 5	0.069	280 55′	3	353
5 — 5	0.661	29 42	3	132
6 — 5	0.479	0 6	73	263 42′
7 — 5	0.219	332 11	62	237 18
8 — 5	0.120	121 32	54	22 5
9 — 5	0.145	90 5	68	355 15
10 — 5	0.129	59 45	70	326 14
11 — 5	0.211	39 34	166	300 19
12 — 5	0.241	213 4	181	108 0
2 — 6	0.001	73		
3 — 6	0.003	194	1	333
4 — 6	0.006	200	2	286
5 — 6	0.038	356 15	3	86
6 — 6	0.251	106 44	3	215
7 — 6	0.200	78 29	30	346
8 — 6	0.092	50 55	24	312
9 — 6	0.047	199 40	19	105
10 — 6	0.052	169 12	12	65
11 — 6	0.026	135 9	7	39
12 — 6	0.013	103 13	4	24
5 — 7	0.003	298	2	343
6 — 7	0.021	72 38	3	155
7 — 7	0.099	183 15		
8 — 7	0.086	156 22	13	60
9 — 7	0.045	130 8	12	34
10 — 7	0.017	275 7	11	177
11 — 7	0.023	242 23	11	153
12 — 7	0.010	219	5	114
6 — 8	0.002	25		
7 — 8	0.011	152		
8 — 8	0.041	260 19	1	135
9 — 8	0.040	233 52	5	138
10 — 8	0.023	205 38	5	109
11 — 8	0.007	352	5	256
12 — 8	0.011	325	5	225
8 — 9	0.006	227		
9 — 9	0.017	336		
10 — 9	0.019	313	1	217
11 — 9	0.011	286	2	195

χ	k_0	K_0	k_1	K_1
g' g				
12 — 9	0″.002	57°	0″.0001	346°
9 — 10	0.003	302		
10 — 10	0.007	50		
11 — 10	0.009	29		
12 — 10	0.006	2		
10 — 11	0.001	20		
11 — 11	0.003	125		
12 — 11	0.004	106		
11 — 12	0.001	97		
12 — 12	0.002	195		
g'' g'				
1 + 1	0.021	179 15′	11	20
1	0.926	145 45	111	322 51′
1 — 1	8.036	79 2.1	20	280 47
1 — 2	0.153	99 26	68	201 39
1 — 3	0.004	97	3	213
2 + 1	0.002	153	1	270
2	0.113	139 36	44	246 39
2 — 1	7.682	354 17.1	979	216 34
2 — 2	12.380	336 43.3	54	113 4
2 — 3	0.235	330 22	110	98 45
2 — 4	0.007	330	6	90
3 + 1	0.001	305		
3	0.060	306 36	189	200 8
3 — 1	28.520	321 46 31″	3917	182 56.8
3 — 2	23.356	119 19 46	1437	307 48
3 — 3	1.372	66 35	192	246 2
3 — 4	0.044	50 6	17	202 38
3 — 5	0.002	45		
4	0.001	284		
4 — 1	0.054	288 22	3	123
4 — 2	0.912	83 39	128	267 4
4 — 3	0.703	18 8	52	203 20
4 — 4	0.257	129 39	9	148
4 — 5	0.014	111	4	256
4 — 6	0.001	106		
5 — 1	0.003	242		
5 — 2	0.297	48 8	64	231 28
5 — 3	0.429	341 6	60	164 40
5 — 4	0.140	92 57	5	270
5 — 5	0.072	207 39	2	207
5 — 6	0.006	187	1	315

χ	k_0	K_0	k_1	K_1
g'' g'				
6—2	0".119	4° 38'	0".0032	191° 48'
6—3	0.244	124 25	50	309 0
6—4	0.055	61 29	9	245
6—5	0.043	172 12	2	0
6—6	0.023	284 39		
6—7	0.002	263		
7—3	0.016	89 21	5	270
7—4	0.019	22 15	4	207
7—5	0.015	135 29	1	315
7—6	0.016	250 29		
7—7	0.008	1		
7—8	0.001	340		
8—3	0.007	53		
8—4	0.011	347		
8—5	0.005	98		
8—6	0.006	214		
8—7	0.006	328		
8—8	0.003	77		
9—4	0.003	131		
9—5	0.001	73		
9—6	0.002	177		
9—7	0.002	290		
9—8	0.002	45		
9—9	0.001	153		
10—7	0.001	256		
10—8	0.001	9		
10—9	0.001	124		
g''' g'				
1 + 1	0.002	270		
1	0.101	287 29		
1—1	1.717	312 59		
1—2	0.027	309 1		

χ	k_0	K_0	k_1	K_1
g''' g'				
1—3	0".001	303°		
2	0.012	269 30'		
2—1	0.904	84 44		
2—2	1.052	86 17		
2—3	0.026	87 22		
2—4	0.001	90		
3—1	0.031	166 12		
3—2	0.103	197 18		
3—3	0.093	39 58		
3—4	0.004	41 59		
4—1	0.001	284		
4—2	0.009	308		
4—3	0.010	151		
4—4	0.015	353		
4—5	0.001	354		
5—2	0.001	67		
5—3	0.001	262		
5—4	0.002	102		
5—5	0.003	308		
6—6	0.001	261		
♀—♄	0.038	0		
♁—♄	0.066	0		
g' g g''				
2—1 + 1	0.022	270		
3—1—1	0.168	288 21		
3—1—2	0.207	79 43		
4—2 + 3	0.063	213 2		
4—1—4	0.106	37 11		
5—2—3	1.884	208 34	0".0294	80° 14'
6—2—3	28.917	6 55.9	7830	242 4
7—2—6	0.153	353 26		

Perturbations of Saturn: Common log $\frac{r'}{r'}$. (In units of the 7th decimal.)

χ	k_0	K_0	k_1	K_1	k_2	K_2	k_3	K_3
g' g								
	+ 1825.0		+ 42.00		+ 0.673		—0.0005	
1	187.3	295° 24'.7	2831.85	57° 59' 22".6	18.924	319° 16'.7	196	168° 20'
2	49.9	293 9	78.37	58 39.0	1.870	302 37	55	197 25
3	14.2	271 43	3.11	70 28	119	300 5	4	180
4	0.6	311	17	64 25	8	299		
—3—1	0.2	111						
—2—1	4.6	165 26	22	263 45				
—1—1	10.4	140 34	36	235 18	1	180		
—1	82.0	110 49	1.15	219 39	20	299		
1—1	3780.8	79 45 7"	3.19	304 46	10	30		
2—1	2442.1	176 2 33	21.60	121 29.5	204	36 17		
3—1	241.2	305 54.3	6.21	207 37	58	187 32		
4—1	35.1	342 36	45	126 52	6	134		
5—1	0.7	309	8	214				
6—1	0.1	294						
—2—2	0.1	158						
—1—2	1.8	241 3	9	341				
—2	3.7	210 18	11	316				

χ	k_0	K_0	k_1	K_1	k_2	K_2
g' g						
1 — 2	55.2	98° 52′	0.26	257° 19′	0.002	189°
2 — 2	643.5	156 34.5	32	14 5	3	0
3 — 2	420.9	141 57.7	11.31	46 59	50	338 15′
4 — 2	7001.9	277 15 14″	170.48	179 38.3	2.255	86 4.7
{ 5 — 2 − 88″.928t }	1141.0	62 49 27	4.36	242 49	75	6 15
6 — 2	18.3	77 17	19.45	306 10	60	202 11
7 — 2	0.6	114	1.06	306 55	6	185
8 — 2			6	307		
−1 — 3	0.1	224	1	303		
— 3	0.8	318 32	4	58		
1 — 3	1.0	46 9	4	61		
2 — 3	5.3	178 39	5	342		
3 — 3	147.1	233 55.8	4	82	1	0
4 — 3	102.0	206 23.6	1.36	107 1	10	11 30
5 — 3	59.7	177 52	1.80	78 59	23	343 19
6 — 3	17.3	178 3	2.86	51 41	48	314 0
7 — 3	34.6	32 39	2.39	340 42	4	254
8 — 3	4.9	210 27	39	153 48	5	61
9 — 3	0.7	275	2	139		
— 4	0.1	298				
1 — 4	0.4	43	2	134		
2 — 4	0.5	17	2	115		
3 — 4	2.8	229 44	1	8		
4 — 4	44.5	311 30	1	122		
5 — 4	31.5	285 3	42	184 23	3	98
6 — 4	14.9	259 21	35	157 31	4	67
7 — 4	8.1	37 4	52	302 47	3	37
8 — 4	21.5	15 52	91	284 18	5	204
9 — 4	93.1	163 39	8.55	67 13	116	331 16
10 — 4	11.0	306 25	1.81	215 3	33	118 27

χ	k_0	K_0	k_1	K_1
g' g				
11 — 4	0.2	102°	0.02	17°
2 — 5	0.2	113	1	214
3 — 5	0.2	106	1	199
4 — 5	1.5	296 38′	1	0
5 — 5	15.6	28 45	1	184
6 — 5	11.9	3 7	16	263
7 — 5	5.5	337 29	14	240
8 — 5	2.7	116 6	17	24
9 — 5	3.7	118 23	21	354
10 — 5	2.7	63 43	18	325
11 — 5	3.6	36 26	26	300
12 — 5	0.7	263	7	113
3 — 6	0.1	191		
4 — 6	0.1	191		
5 — 6	0.8	8 29	1	69
6 — 6	5.9	105 37		
7 — 6	5.0	80 27	7	341
8 — 6	2.4	56 33	6	317
9 — 6	1.0	191 57	6	96
10 — 6	1.3	171 26	6	74
11 — 6	0.7	144	3	47

χ	k_0	K_0	k_1	K_1
g' g				
12 — 6	0.1	120°	0.02	15°
5 — 7	0.1	279		
6 — 7	0.4	83		
7 — 7	2.4	182 4′		
8 — 7	2.2	158 50	3	59
9 — 7	1.1	135 6	3	35
10 — 7	0.4	265	2	171
11 — 7	0.6	247	3	150
12 — 7	0.3	222	2	122
7 — 8	0.2	158		
8 — 8	1.0	258		
9 — 8	1.0	236	2	140
10 — 8	0.6	214	2	113
11 — 8	0.1	334	1	252
12 — 8	0.2	325	1	228
8 — 9	0.1	231		
9 — 9	0.4	333		
10 — 9	0.5	313	1	216
11 — 9	0.3	293	1	191
9 — 10	0.1	304		
10 — 10	0.2	48		

χ	k_0	K_0	k_1	K_1
g' g				
11 — 10	0.2	29°		
12 — 10	0.2	8		
11 — 11	0.1	122		
12 — 11	0.1	105		
g'' g'				
1 + 1	0.3	356	0.01	188°
1	3.2	345 3′	8	140
1 — 1	59.0	79 3	1	277
1 — 2	2.5	95 57	6	202
1 — 3	0.1	95		
2	1.0	328 4	6	59
2 — 1	35.5	350 35	28	217 25′
2 — 2	154.1	336 43.3	5	106
2 — 3	5.3	332 13	10	98
2 — 4	0.2	332	1	90
3	0.6	126	18	20 33
3 — 1	26.4	137 55	16	355 1
3 — 2	237.4	119 5.6	1.43	308 4
3 — 3	22.1	69 58	17	252 12
3 — 4	1.1	57 13	2	211
3 — 5	0.1	52		
4 — 1	0.4	104		
4 — 2	6.7	80 4	9	266
4 — 3	9.7	19 51	7	202
4 — 4	4.4	128 12	1	158
4 — 5	0.3	115		
5 — 2	1.1	38 31	2	225
5 — 3	5.2	342 27	7	166
5 — 4	2.2	93 59	1	270
5 — 5	1.3	206 15		
5 — 6	0.1	190		
6 — 2	0.2	172		
6 — 3	2.4	123 27	5	307
6 — 4	0.8	65 50	2	256
6 — 5	0.7	173 49		

χ	k_0	K_0	k_1	K_1
g'' g'				
6 — 6	0.4	283°		
6 — 7	0.1	266		
7 — 3	0.1	84		
7 — 4	0.3	26	0.01	327°
7 — 5	0.3	139		
7 — 6	0.3	252		
7 — 7	0.2	359		
8 — 4	0.1	348		
8 — 5	0.1	104		
8 — 6	0.1	218		
8 — 7	0.1	329		
8 — 8	0.1	75		
g''' g'				
1	0.2	337		
1 — 1	15.4	312 58′		
1 — 2	0.5	310		
2	0.2	86		
2 — 1	7.6	84 55		
2 — 2	14.8	86 17		
2 — 3	0.6	87		
3 — 1	0.2	173		
3 — 2	1.3	195 39		
3 — 3	1.5	40 12		
3 — 4	0.1	42		
4 — 2	0.1	307		
4 — 3	0.1	148		
4 — 4	0.3	353		
5 — 5	0.1	307		
♀ — ♄	0.8	0		
♅ — ♄	1.4	0		
g' g g''				
5 — 2 — 3	19.8	208 34	31	80 14′
6 — 2 — 3	8.4	2 4		

Perturbations of Saturn: $\Delta\beta'$.

χ	k_0	K_0	k_1	K_1
g' g				
	—0″.329		—0″.0109	
2	0.204	287° 13′	19	231°
3	0.019	269	3	162
4	0.005	51		
5	0.002	331		
— 3 — 1	0.003	209		
— 2 — 1	0.002	41		
— 1 — 1	0.026	37	20	311
— 1	1.803	116 2	245	32 22′
1 — 1	0.841	210 40	138	163 9
2 — 1	2.905	225 28.4	482	310 59
3 — 1	0.721	185 4	18	276
4 — 1	0.057	301 28	2	117
5 — 1	0.037	310 15	2	27
6 — 1	0.001	340		

χ	k_0	K_0	k_1	K_1
g' g				
— 2 — 2	0″.001	279°		
— 1 — 2	0.002	81	0″.0002	207°
— 2	0.063	91 47′	4	237
1 — 2	0.258	11 58	29	299 18′
2 — 2	0.116	319 33	8	90
3 — 2	0.215	207 35	54	197 9
4 — 2	8.679	277 12.5	155	66 57
5 — 2	0.370	111 9	56	329 47
6 — 2	0.245	16 42	75	269 18
7 — 2	0.011	19	9	249
— 1 — 3	0.001	352		
— 3	0.003	114		
1 — 3	0.007	84		
2 — 3	0.087	89 53		
3 — 3	0.041	53 10		

χ	k_0	K_0	k_1	K_1	χ	k_0	K_0	χ	k_0	K_0
g' g					g' g			g'' g'		
4 — 3	0″.077	199° 39′			4 — 6	0″.001	237°	3 — 4	0″.003	64°
5 — 3	0.117	176 9			5 — 6	0.005	323	4 — 1	0.005	208
6 — 3	0.096	155 49			6 — 6	0.004	292	4 — 2	0.025	349 4′
7 — 3	0.048	300 27			7 — 6	0.001	63	4 — 3	0.023	281 40
8 — 3	0.002	247			8 — 6	0.001	21	4 — 4	0.001	331
9 — 3	0.001	225			9 — 6	0.001	8	5 — 1	0.001	165
2 — 4	0.003	139			10 — 6	0.001	351	5 — 2	0.003	333
3 — 4	0.033	167 31			6 — 7	0.002	38	5 — 3	0.021	244 54
4 — 4	0.018	134 26			7 — 7	0.002	9	5 — 4	0.005	341
5 — 4	0.014	266 3			g'' g'			6 — 2	0.001	232
6 — 4	0.013	246 33			1 + 1	0.019	259 21′	6 — 3	0.012	32
7 — 4	0.011	230			1	0.080	220 17	6 — 4	0.003	333
8 — 4	0.002	171			1 — 1	0.036	11 34	6 — 5	0.001	69
9 — 4	0.087	161 51	0″.0012	250°	1 — 2	0.035	298 43	7 — 3	0.001	0
10 — 4	0.009	341			1 — 3	0.002	306	7 — 4	0.001	288
11 — 4	0.002	273			2 + 1	0.003	164	7 — 5	0.001	45
3 — 5	0.001	189			2	0.040	152 57	g''' g'		
4 — 5	0.013	245			2 — 1	0.110	301 20	1 + 1	0.002	137
5 — 5	0.009	214			2 — 2	0.031	277 36	1	0.005	146
6 — 5	0.004	349			2 — 3	0.008	2	1 — 1	0.001	4
7 — 5	0.003	317			3 + 1	0.002	294	1 — 2	0.002	120
8 — 5	0.002	303			3	0.032	289 10	2	0.003	276
9 — 5	0.003	272			3 — 1	0.046	221 32	2 — 1	0.018	98 27
10 — 5	0.002	247			3 — 2	0.599	20 3	2 — 2	0.001	111
11 — 5	0.002	219			3 — 3	0.037	17 4	3 — 2	0.004	232

$$
\begin{aligned}
\sin \beta_0' = {} & \sin i' \sin (l' - \Omega') \\
& + 82''.2723\, T \sin (l' + 346^\circ 53' 28''.65) \\
& + 0''.42282\, T^2 \sin (l' + 75^\circ 31'.9) \\
& + 0''.001422\, T^3 \sin (l' + 163^\circ 37').
\end{aligned}
$$

Reduction of orbit longitudes to the mean equinox and ecliptic of date

$$
\begin{aligned}
= {} & + 97''.774 \sin (2l' + 315^\circ 18' 21''.9) + 0''.023 \sin (4l' + 270^\circ 37') \\
& + [5026''.6850 + 1''.7921 \sin (2l' + 54^\circ 33'.2) + 0''.0008 \sin (4l' + 7^\circ)]\, T \\
& + [1''.10463 + 0''.01737 \sin (2l' + 148^\circ 11') + 0''.00002 \sin (4l' + 90^\circ)]\, T^2 \\
& + [0''.000166 + 0''.000115 \sin (2l' + 239^\circ 38')]\, T^3 \\
& + [-0''.0000488 + 0''.0000005 \sin (2l' + 338^\circ)]\, T^4 - 0''.00000023\, T^5 .
\end{aligned}
$$

MEMOIR No. 41.

A Reply to Mr. Neison's Strictures on Delaunay's Method of Determining the Planetary Perturbations of the Moon.

(Monthly Notices of the Royal Astronomical Society, Vol. XLVII, pp. 1-8, 1886.)

For several years past Mr. Neison has been maintaining in the *Monthly Notices* and *Memoirs* of the Society that Delaunay's investigation of the two long period inequalities in the Moon's motion arising from the action of *Venus* is seriously defective, on account of the omission by him of a certain class of terms. In the *Monthly Notices* for last June there appears a long article by him upholding this view; to this I wish more especially to direct attention.

At the outset I may be allowed to say that all this criticism is without foundation. It appears to arise, partly from the very confused conception Mr. Neison seems to have of the nature of Delaunay's method, and partly because he fails to notice that Delaunay, after setting the degree of approximation he wishes to attain, always rigorously adheres to it. If we were obliged to admit the validity of *all* the statements in this article, an easy corollary from them would be that Lagrange's general method of the variation of arbitrary constants in the problems of mechanics was a blunder. Now, I think that no one acquainted with this method could, for a moment even, entertain such a proposition. Hence we may conclude there is some flaw in the reasoning of this Paper. But this must be substantiated by noticing *seriatim* the objectionable points.

In the first place, why bring forward Hansen's published values of the coefficients of these inequalities for the purpose of throwing discredit upon Delaunay's values, when their author, himself, virtually confesses he has no confidence in them, by saying he had computed them in two different ways, and found essentially different results? And, to the very end of his life, he appears never to have been able to find out whether one of these results was right, and which it was, or whether both were wrong. It would be an amusing circumstance should it turn out that the set of values, withheld from publication by Hansen, were identical with those of Delaunay.

There is some inexactitude in Mr. Neison's statement, regarding the degree of approximation adopted by Delaunay in calculating the coefficient

whose argument is $8l'' - 13l'$. In this connection we note that, on account of the close proximity of the Moon to the Earth, a planet cannot produce in her motion inequalities of the same order with those it produces in the Earth, but that they are only of the order of these multiplied by the solar disturbing force; and this is as true of the indirect action as of the direct. Now, on referring to Delaunay's work, we see that he has considered, not only the term of the lowest order in each portion of the coefficient, but also multiples of this by m^2. Hence, it is correct to say that he has considered terms of the order of the mass of *Venus* multiplied by the square of the solar disturbing force, but not those multiplied by the cube of the latter.

Mr. Neison regards the evidence adduced in his earliest Paper, as conclusively establishing the omission by Delaunay of a certain class of terms. But what was this evidence? Simply, that Hansen was at variance with Delaunay. Now, since Hansen was as much at variance with himself as he was with Delaunay, what weight ought to be attributed to this evidence? Then, Mr. Neison believes that certain discrepancies between results, obtained on the one hand by himself, and, on the other, by M. Gogou and myself, have their origin in the same cause. Now, if Hansen's investigation and also Mr. Neison's were accessible, this point could be immediately pronounced upon; but since they are not, it appears useless to speculate on the matter.

Mr. Neison says (p. 416), "He (Delaunay) substitutes the preceding value for the term in the disturbing function with the argument ζ in the differential equation and integrates." This does not correctly represent what Delaunay does. For he substitutes in the differential equation not only the term factored by $\cos\zeta$, but also the non-periodic portion of R: to wit, in the memoir of 1862, the terms

$$\frac{\mu}{2a} + m'\frac{a^2}{a'^3}\left[\tfrac{1}{4} + \tfrac{3}{8}e^2 + \tfrac{225}{64}e^2\frac{n'}{n} - \left(\tfrac{81}{32} - \tfrac{971}{32}e^2\right)\frac{n'^2}{n^2}\right],$$

and, in the memoir of 1863, the terms

$$\frac{\mu}{2a} + m'\frac{a^2}{a'^3}\left[\tfrac{1}{4} - \tfrac{3}{2}\gamma^2 + \tfrac{3}{8}e^2 + \tfrac{3}{8}e'^2\right].$$

The terms differ in the two cases, because the degree of approximation aimed at requires the preservation of different terms with allowable neglect of all the rest. From this non-periodic portion of R results, in both cases, much the larger part of the two inequalities considered. Mr. Neison's failure to note this completely invalidates his argument on the two following pages, by which he attempts to prove the incompleteness of Delaunay's

procedure. And, in this connection, it may be noted that it is not necessary that the coefficient in (38) should vanish identically in order to prove Delaunay right; it is necessary only that it should turn out of such an order of smallness as to prove that the adopted degree of approximation had been attained.

On p. 417 it is said that the coefficients B and B' "only differ by small quantities, unimportant for the present purpose." So far from this being the case, the difference $B' - B$ constitutes one portion of the terms which Mr. Neison, all along, has been asserting were neglected by Delaunay.

That Delaunay, in treating the two Venus inequalities, discarded his own method, and employed the old one recommended by Poisson, is erroneously stated on p. 423. The fact is that the method followed is the same as that he had used in deriving the solar perturbations. Next, Delaunay is found fault with (p. 424) because he confines himself to calculating in R the term which has the argument of the particular inequality he is dealing with; while it is plain that there are a multitude of terms in R, having other arguments, which could contribute to the value of the coefficient sought. This is true, but Delaunay's reasons for passing by these terms are quite evident. In the first place, it must be remembered that his final expression for the inequality is a formula of substitution, which must be made, not only in the mean longitude of the Moon, but also in the equation of the centre, in the evection, variation and in all the inequalities arising from solar action. Hence, Delaunay's method of treatment enables him to obtain, with very little additional labor, all the terms in the expression for the *true* longitude which involve the very small divisor arising from the slow motion of the argument which he is considering; and that whatever may be their arguments. And, secondly, while the terms in R, having other arguments, which would be treated by Delaunay as giving rise each to a distinct transformation, can, in a strict sense, add something to the coefficient of the inequality in the *true* longitude, practically these terms are insensible; for although they may be of the same order, before integration, as the quantities retained, they are altogether independent of the excessively small divisor which arises from the slow motion of the argument of the inequality. As illustrating this point, it may be remarked that, in the case of the two Venus inequalities in question, we get such relatively large coefficients as $16''$ and $0''.27$ only by multiplying the corresponding terms in R by factors which are about 15,000,000 in the first, and 10,000,000 in the second inequality. Hence, if there are other terms, which rigorously ought to be added to the preceding values, but which, while in other respects of the same order of smallness,

have factors not much exceeding unity, it is very apparent they may be neglected.

In the next place we find Delaunay charged with neglecting every term of the solar perturbations save the term of the lowest order in the variation in calculating the proper form for R. And it is said that his development "in no sense depends on his method of transformed elements, though made to appear as if it does; nor does it differ in any way from the values hitherto employed by astronomers save in being somewhat less complete." These statements misrepresent Delaunay. He arranges under four different heads the transformations made by him, and they involve no less than 16 out of the 57 operations of his first volume, besides 4 complementary ones. And whether the amount of work in this be regarded as much or little, I have ascertained that it is precisely sufficient to obtain the degree of approximation he proposes in the coefficients B, viz. to terms involving m^3. Carrying the approximation farther could only have afforded him terms of a higher order. It is, of course, open to Mr. Neison to say he deems this degree of approximation insufficient; and nothing can be said in opposition. But this is very different from saying Delaunay has committed errors. Again, I am not aware of the existence of any published investigation in which the degree of approximation is greater.

The reasoning Mr. Neison employs to show that Delaunay deserts, in this investigation, his own method and returns to the old method recommended by Poisson, is certainly very strange. He notes that the differential equation used has nothing in it to distinguish it from the corresponding one which Poisson would have used. But from what circumstance does this state of things arise? Simply because it is Delaunay's habit to omit, in the statement of his equations, every term which gives rise, in the final result, only to terms of a higher order than he has agreed to retain. The factors in question, in Delaunay's method can be expressed only as infinite series; it is necessary, therefore, to cut them off at some point, and he determines this point in the way just stated. If reference is made to the same equation, in the memoir where Delaunay treats the other Venus inequality, it will be found to be duly distinguished by the presence of additional terms, Delaunay writing as many as are just sufficient for his purpose.

Mr. Neison next notices two assumptions, which he says have been made by Delaunay in his integration.

The first is that the factor $\frac{2}{an}$, which multiplies $\frac{dR}{dl}$, is treated as if it were constant. But here he forgets that, with Delaunay, at this stage of

the work, the symbols a, e, γ, l, g, and h, denote quantities which have no solar perturbations; and that, consequently, the deviation of $\frac{2}{an}$ from a constant has the mass of the planet as a factor. Thus, as $\frac{dR}{dl}$ already has this factor, the additional terms, which would in this manner arise, would have the square of the mass of the planet as factor; these, as all other investigators, Delaunay expressly neglects.

With regard to the second assumption, in reference to which Mr. Neison makes what he thinks his chief point against Delaunay, let us consider what is the essential difference between Delaunay's method and that employed by the earlier investigators. Delaunay said to himself, Do not let us go back to the elements of the Keplerian ellipse every time we have to consider the action of a new force on the Moon, but let us determine our new wave of motion in such a way that it may be superposed on the curve which the Moon would describe under the action of all the forces previously considered, instead of on the Keplerian ellipse. At any stage of progress, in expressing the Moon's co-ordinates, there must, of necessity, appear in them six arbitrary constants which have been introduced by integration. Let us take these as variables, instead of the six elements of the Keplerian ellipse. This course demands that the differential equations employed by the earlier investigators should be somewhat modified. The modification appears as a change in the values of the quantities which Poisson denoted generally by the symbol $[a, b]$. Now, just as it would be absurd to maintain that the elements of the Keplerian ellipse suffer perturbations from the action of a centrobaric Earth, so it is absurd to maintain that the quantities a, e, γ, l, g, and h, employed by Delaunay after he has got through with the solar perturbations and has arrived at the treatment of the planetary perturbations, and which are the elements of the curve which would be described by the Moon under the combined action of the Earth and Sun, suffer perturbations from the latter body. Yet Mr. Neison's argument, when divested of its obscurities, is seen to be nothing more or less than a plea that these quantities do suffer perturbations from the Sun.

To make the matter plainer, let us suppose that Delaunay, groping about in the dark, had fallen upon the Poissonian equations, and, thinking them to be his own, had used them as such; and, moreover, on making his substitutions, had made them only in the elliptic portion of the co-ordinates. Then he would have committed the very error Mr. Neison lays to his charge. But since he uses equations suitably modified to the new signification of the

quantities a, e, etc., and, moreover, makes his substitutions in the complete expressions for the Moon's co-ordinates, and not in the elliptic portion only, as the earlier investigators do, is it not plain that, by these two modifications, he obtains terms which he would not have obtained in the former supposed case? Now these terms, in sum, are precisely equivalent to those Mr. Neison accuses him of neglecting by omitting to include R''' in his disturbing function. Thus it is seen that Delaunay takes account of R''' in an indirect manner, the peculiar nature of his method absolving him from considering the terms arising from R''' as a separate class.

Perhaps the matter will be clearer still if we say that, just, as in determining the solar perturbations we have no class of terms of the order of the product of the mass of the Earth by the mass of the Sun, simply because the Earth's action is considered as the principal force, so when we come to treat the planetary perturbations by Delaunay's method, there is no special class of terms of the order of the product of the Sun's mass by the planet's mass, for the reason that here the combined actions of the Earth and Sun are regarded as forming the principal force.

Next we must not pass over without notice the quite erroneous method. Mr. Neison proposes (pp. 430, 431) for getting the proper expressions for the Poissonian quantities $[a, b]$; viz. by substituting for the elements in the expressions proper to the older form of the differential equations their complete values as functions of the time, and then neglecting all the periodic terms. It is very certain this procedure will not give the same values as Delaunay has, who obtains them by taking the partial derivatives of a, e, and γ with respect to the elements L, G, and H, which are the conjugates of l, g, and h.

Mr. Neison is not content with what he has already said to establish the serious imperfection of Delaunay's method, but fortifies himself in the belief of it by a new line of argument (pp. 432-437), where he gives his conception of the essential nature of Delaunay's transformations. But his argument is fatally vitiated because he will have it that the transformations in question are rigorously linear in their operation. Thus, to illustrate, suppose Delaunay has

Operation 1.

Replace a_0 by $a_1 + f_1(a_1, e, \text{etc.})$.

Operation 2.

Replace a_1 by $a_2 + f_2(a_2, e_2, \text{etc.})$.

(I use the subscripts, which Delaunay has not, that my meaning may be clear.) According to Mr. Neison's way of looking at things, these two operations are equivalent to

$$\text{Replace } a_0 \text{ by } a_2 + f_1(a_2, e_2, \text{etc.}) + f_2(a_2, e_2, \text{etc.}).$$

Thus he fails to see that Delaunay intends the a_1, under the functional sign f_1, to be eliminated by the substitution of Operation 2, as well as the a_1 which is outside of it. In consequence he misses all the terms which are of the order of the product of f_1 by f_2.

Now, suppose that f_1 belongs to an operation which is concerned with solar perturbations, and f_2 to one concerned with planetary perturbations. Then Mr. Neison, by his erroneous interpretation of Delaunay's processes, fails to get some terms of the order of the product of the masses of the Sun and planet, which, nevertheless, Delaunay has. Now, these are the very terms Delaunay is accused of neglecting. And, what is sufficiently singular, Mr. Neison appears to regard the symbols a, e, etc., which are under the functional signs f_1, f_2, etc., as having every where throughout the whole series of operations the same signification, and as being absolute constants; so that, for him, all the f's are explicit functions of the time.

There is another way in which Mr. Neison's error may be illustrated. Suppose we write one of the differential equations of the Moon's motion in rectangular co-ordinates, thus

$$\frac{d^2x}{dt^2} - \frac{d\Omega_0}{dx} = \frac{dR^{(0)}}{dx} + e'\frac{dR^{(1)}}{dx} + e'^2\frac{dR^{(2)}}{dx} + \ldots + \beta\frac{dR_0}{dx} + m''\frac{dR_1}{dx} + \text{etc.},$$

where Ω_0 denotes the potential of the force exerted by a centrobaric Earth; and the portion of the disturbing function due to solar action has been broken into a number of parts $R^{(0)}$, $e'R^{(1)}$, $e'^2R^{(2)}$, etc., severally proportional to the various powers of the solar eccentricity e'; and βR_0 is the portion due to the figure of the Earth, β being a constant which measures the deviation of the Earth from a centrobaric body; in fine, $m''R_1$, is the portion due to the action of a planet whose mass is m''. Then Delaunay's way of proceeding is very similar to this: he first ascertains what would be the expressions for the Moon's coordinates were $R^{(0)}$ the complete disturbing function, by making variable the a, e, γ, l, g, and h which appear in the elliptic formulæ; he then transposes $R^{(0)}$ over to the left member of the equation, and the potential of the principal force is now no longer Ω_0 but $\Omega_0 + R^{(0)}$; he then proceeds to treat $e'R^{(1)}$ as if it alone constituted the whole of the disturbing function, using the elements a, e, γ, l, g, and h, which stand in his last

expressions for the co-ordinates as variables, not those which belong to the elliptic expressions. When this is done, $e'R^{(1)}$ is transferred to the left member, and the potential of the principal force is now $\Omega_0 + R^{(0)} + e'R^{(1)}$, and the work is continued as before.

Now, Mr. Neison admits the legitimacy of all this as long as we are dealing with the portions of the disturbing function which arise from solar action; but says that, the moment we arrive at the term $m''R_1$, all changes. Then certain ghosts, as it were, of the portions $R^{(0)}$, $e'R^{(1)}$, etc., unbidden return to the right member and trouble the portion $m''R_1$. Thus we have the strange spectacle of forces figuring at once as principal and as disturbing. Mr. Stockwell made a precisely similar objection to my elaboration of the inequalities due to the figure of the Earth, which was disposed of by Prof. Adams in a single sentence.

If all this be true, what becomes of the assertion, often reiterated, that when the differential equations are written down, all the rest is a pure question of analysis? On Mr. Neison's and Mr. Stockwell's view, the analyst, who does the integrating, needs an astronomical or mechanical prompter at his elbow to inform him of the exact physical import of the constants β or m'', otherwise he will infallibly go wrong.

MEMOIR No. 42.

Coplanar Motion of Two Planets, One Having a Zero Mass.

(Annals of Mathematics, Vol. III, pp. 65-73, 1887.)

The supposition that two planets circulate about their central body in the same plane enables us to dispense with two differential equations of the second order in the general problem of three bodies. The further supposition, that the mass of one of them is too insignificant to have any sensible effect on the motion of the other, enables us to consider the motion of the latter as known and as taking place according to the laws of Kepler. Hence, in this case, the two co-ordinates of the planet of zero mass are the only unknowns; aud they are given by two differential equations of the second order. These suppositions have, approximately, place in several cases in the solar system, but I have more especially in view the motion of the satellite Hyperion as disturbed by the action of Titan. My object in this paper is simply to point out a method of proceeding, which may, I think, be advantageously employed in this case.

Employing the usual notation x, y, r, for the rectangular co-ordinates and radius vector of the planet whose motion is to be determined, x', y', r', for the corresponding quantities belonging to the acting planet, m' the mass of the latter, and M the mass of the central body, the differential equations of motion will be

$$\frac{d^2x}{dt^2}=\frac{\partial\Omega}{\partial x}, \quad \frac{d^2y}{dt^2}=\frac{\partial\Omega}{\partial y},$$

where Ω, the potential function, has the following expression:

$$\Omega=\frac{M}{\sqrt{(x^2+y^2)}}+m'\left[\frac{1}{\sqrt{[(x-x')^2+(y-y')^2]}}-\frac{x'x+y'y}{r'^3}\right].$$

The co-ordinates of m' satisfy the differential equations

$$\frac{d^2x'}{dt^2}+\frac{M+m'}{r'^3}x'=0, \quad \frac{d^2y'}{dt^2}+\frac{M+m'}{r'^3}y'=0.$$

We can, without any loss of generality, assume that the axis of x is directed toward the lower apsis of m'. Then the integrals of the last-stated differential equations are

$$x'=a'(\cos\epsilon'-e'), \quad y'=a'\sqrt{(1-e'^2)}\sin\epsilon',$$

where ε' is derived from the equation

$$n't + c' = \varepsilon' - e' \sin \varepsilon',$$

a', e', c' being constants, and n' being the equivalent of $\sqrt{\left(\frac{M + m'}{a'^3}\right)}$.

It is desirable to know what the differential equations determining x and y become when expressed in terms of any other variables. For this end Lagrange's canonical form of the equations serves very conveniently. Let the new variables be u and s, and employ the subscript $(_1)$ to denote the complete differential co-efficient with respect to t of any variable to which it is attached. Then T standing for $\frac{1}{2}(x_1^2 + y_1^2)$ expressed in terms of u, s, u_1, s_1, Lagrange's canonical form of the equations is

$$\frac{d}{dt}\frac{\partial T}{\partial u_1} - \frac{\partial T}{\partial u} = \frac{\partial \Omega}{\partial u}, \quad \frac{d}{dt}\frac{\partial T}{\partial s_1} - \frac{\partial T}{\partial s} = \frac{\partial \Omega}{\partial s}.$$

As we have

$$x_1 = \frac{\partial x}{\partial u}u_1 + \frac{\partial x}{\partial s}s_1 + \frac{\partial x}{\partial t},$$

$$y_1 = \frac{\partial y}{\partial u}u_1 + \frac{\partial y}{\partial s}s_1 + \frac{\partial y}{\partial t},$$

we get

$$\begin{aligned} T = \tfrac{1}{2} & \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right] u_1^2 + \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial s} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial s}\right) u_1 s_1 \\ & + \tfrac{1}{2}\left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2\right] s_1^2 + \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial t}\right) u_1 \\ & + \left(\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial s}\frac{\partial y}{\partial t}\right) s_1 + \tfrac{1}{2}\left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2\right]. \end{aligned}$$

It is very plain from the form of Lagrange's equations that if the variables u and s were so assumed that one of them, u for instance, should disappear at once from the expressions for T and Ω, we should have an integral of the problem. For then $\frac{d}{dt}\frac{\partial T}{\partial u_1} = 0$; and, integrating, $\frac{\partial T}{\partial u_1} =$ a constant. This selection, in a theoretical sense, is always possible, and in as many essentially distinct ways as there are first integrals of the problem, which, in the present case, are four, but although it is easy in innumerable ways, to make Ω depend on one variable, it is not so easy to make the six factors of the general expression for T depend solely on the same variable. And, when we inquire what equations must be satisfied for this, we find that they are essentially the same as those which are satisfied by the Eulerian multipliers. Hence, nothing is gained by approaching the problem from this side.

I propose to take u and s so that

$$x = \rho x'u + \rho y's, \quad y = \rho y'u - \rho x's,$$

where ρ denotes a function of t supposed known, but, for the present, left indeterminate. From these equations may be derived

$$r^2 = \rho^2 r'^2 (u^2 + s^2), \quad x'x + y'y = \rho r'^2 u.$$

Hence the potential function, in terms of u and s, becomes

$$\Omega = \frac{1}{\rho r'}\left[\frac{M}{\sqrt{(u^2+s^2)}} + \frac{m'}{\sqrt{[(u-\rho^{-1})^2+s^2]}} - m\rho^3 u\right].$$

In the general expression for T we substitute the values

$$\frac{\partial x}{\partial u} = \rho x', \quad \frac{\partial y}{\partial u} = \rho y', \quad \frac{\partial x}{\partial s} = \rho y', \quad \frac{\partial y}{\partial s} = -\rho x',$$

$$\frac{\partial x}{\partial t} = \frac{d(\rho x')}{dt}u + \frac{d(\rho y')}{dt}s, \quad \frac{\partial y}{\partial t} = \frac{d(\rho y')}{dt}u - \frac{d(\rho x')}{dt}s.$$

The result is

$$T = \tfrac{1}{2}\rho^2 r'^2 (u_1^2 + s_1^2) - a'^2 n' \sqrt{(1-e'^2)}\,\rho^2 (us_1 - su_1) + \tfrac{1}{2}\frac{d(\rho^2 r'^2)}{dt}(uu_1 + ss_1)$$
$$+ \tfrac{1}{2}\left[a'^2 n'^2\left(\frac{2a'}{r'} - 1\right)\rho^2 + 2r'\frac{dr'}{dt}\rho\frac{d\rho}{dt} + r'^2\frac{d\rho^2}{dt^2}\right](u^2+s^2).*$$

For the sake of brevity we may write, h_1, h_2, h_3, h_4 being known functions of t,

$$T = \tfrac{1}{2}h_1(u_1^2 + s_1^2) - h_2(us_1 - su_1) + \tfrac{1}{2}h_3(u^2+s^2) + h_4(uu_1 + ss_1).$$

This, substituted in Lagrange's canonical form of the differential equations, gives as the equations of the problem,

$$\frac{d}{dt}\left(h_1\frac{du}{dt}\right) + 2h_2\frac{ds}{dt} + \left(\frac{dh_4}{dt} - h_3\right)u + \frac{dh_2}{dt}s = \frac{\partial\Omega}{\partial u},$$

$$\frac{d}{dt}\left(h_1\frac{d}{dt}\right) - 2h_2\frac{du}{dt} - \frac{dh_2}{dt}u + \left(\frac{dh_4}{dt} - h_3\right)s = \frac{\partial\Omega}{\partial s}.$$

Let us now adopt a more general independent variable than the time. Calling this ζ, let $dt = \theta d\zeta$, in which θ may be regarded as a function either of t or ζ. The second supposition will be the more advantageous. In either case as we obtain, on integrating, u and s as functions of ζ, it will be necessary to have the values of ζ which correspond to given values of the time,

*For pointing out an error which exists in the original memoir in this equation, and whose influence vitiated some of the following equations, I am indebted to Prof. G. H. Darwin.

and thus the inverse function will have to be considered. Then, in terms of the new independent variable,

$$\frac{d}{d\zeta}\left(\frac{h_1}{\theta}\frac{du}{d\zeta}\right) + 2h_3\frac{ds}{d\zeta} + \left(\frac{dh_4}{d\zeta} - \theta h_5\right)u + \frac{dh_3}{d\zeta}s = \frac{\partial(\theta\Omega)}{\partial u},$$
$$\frac{d}{d\zeta}\left(\frac{h_1}{\theta}\frac{ds}{d\zeta}\right) - 2h_3\frac{du}{d\zeta} - \frac{dh_3}{d\zeta}u + \left(\frac{dh_4}{d\zeta} - \theta h_5\right)s = \frac{\partial(\theta\Omega)}{\partial s}.$$

We can now consider how ρ and θ should be assumed in order that the differential equations may be most simplified. In the first place it appears important that the potential function Ω should be freed from the independent variable ζ. This is accomplished by putting $\rho = 1$. In the second place it seems we cannot readily do better than take the eccentric anomaly ε' of the attracting planet as the independent variable ζ. Then

$$dt = \frac{r'}{a'n'}d\varepsilon', \text{ and } \theta = \frac{r'}{a'n'}.$$

Also we have

$$h_1/\theta = a'^2n'(1 - e'\cos\varepsilon'),\quad h_3 = a'^2n'\sqrt{(1-e'^2)},\quad \theta h_5 = a'^2n'(1 + e'\cos\varepsilon'),$$
$$\frac{\theta}{\rho r'} = \frac{1}{a'n'},\quad M + m' = a'^3n'^2.$$

For the sake of simplicity let the signification of Ω be changed, and, putting $\dfrac{m'}{M+m'} = \nu$, let

$$\Omega = \frac{1-\nu}{\sqrt{(u^2+s^2)}} + \frac{\nu}{\sqrt{[(u-1)^2+s^2]}} - \nu u.$$

Then our differential equations take the form

$$\frac{d}{d\varepsilon'}\left[(1 - e'\cos\varepsilon')\frac{du}{d\varepsilon'}\right] + 2\sqrt{(1-e'^2)}\frac{ds}{d\varepsilon'} - \left(1 - \frac{e'^2\cos^2\varepsilon'}{1 - e'\cos\varepsilon'}\right)u = \frac{\partial\Omega}{\partial u},$$
$$\frac{d}{d\varepsilon'}\left[(1 - e'\cos\varepsilon')\frac{ds}{d\varepsilon'}\right] - 2\sqrt{(1-e'^2)}\frac{du}{d\varepsilon'} - \left(1 - \frac{e'^2\cos^2\varepsilon'}{1 - e'\cos\varepsilon'}\right)s = \frac{\partial\Omega}{\partial s}.$$

It will be noticed that the potential function Ω is, by this assumption of variables, completely freed from co-ordinates expressing the position of the attracting planet; and that the two factors $1 - e'\cos\varepsilon'$ and $1 + e'\cos\varepsilon'$, very simple functions of the independent variable ε', are the only evidences of the position of this body in the differential equations. And, of the four elements of its orbit, e' is the only one we have to deal with.

We propose now to see whether the introduction of elliptic co-ordinates will bring about any simplification in the problem. Supposing

$$x_1 = s,\quad x_2 = u - \tfrac{1}{2},$$

let
$$\frac{x_1^2}{a_1 + \lambda_1} + \frac{x_2^2}{a_2 + \lambda_1} = 1, \quad\text{and}\quad \frac{x_1^2}{a_1 + \lambda_2} + \frac{x_2^2}{a_2 + \lambda_2} = 1,$$

be the equations of a confocal ellipse and hyperbola, a_1 and a_2 being constants and λ_1 and λ_2 the new variables destined to take the place of u and s. By eliminating x_2^2 from these equations we obtain

$$\frac{a_1 - a_2}{(a_1 + \lambda_1)(a_1 + \lambda_2)} x_1^2 = 1;$$

whence

$$x_1 = \sqrt{\left[\frac{(a_1 + \lambda_1)(a_1 + \lambda_2)}{a_1 - a_2}\right]}.$$

The expression of x_2 in terms of λ_1 and λ_2 is obtained from this by simply interchanging a_1 and a_2. Thus

$$x_2 = \sqrt{\left[\frac{(a_2 + \lambda_1)(a_2 + \lambda_2)}{a_2 - a_1}\right]}.$$

We now proceed to find what Ω becomes in terms of λ_1 and λ_2. By taking the sum of the squares of the last two equations we get

$$x_1^2 + x_2^2 = a_1 + a_2 + \lambda_1 + \lambda_2.$$

Thus far a_1 and a_2 have been left indeterminate, but we now assume

$$a_2 - a = \tfrac{1}{4}.$$

Then

$$\begin{aligned}
u^2 + s^2 &= (x_2 + \tfrac{1}{2})^2 + x_1^2 \\
&= 2a_2 + \lambda_1 + \lambda_2 + 2\sqrt{[(a_2 + \lambda_1)(a_2 + \lambda_2)]} \\
&= [\sqrt{(a_2 + \lambda_1)} + \sqrt{(a_2 + \lambda_2)}]^2, \\
\sqrt{(u^2 + s^2)} &= \sqrt{(a_2 + \lambda_1)} + \sqrt{(a_2 + \lambda_2)}, \\
(u - 1)^2 + s^2 &= (x_2 - \tfrac{1}{2})^2 + x_1^2 \\
&= 2a_2 + \lambda_1 + \lambda_2 - 2\sqrt{[(a_2 + \lambda_1)(a_2 + \lambda_2)]}, \\
\sqrt{[(u - 1)^2 + s^2]} &= \sqrt{(a_2 + \lambda_1)} - \sqrt{(a_2 + \lambda_2)}, \\
u &= 2\sqrt{[(a_2 + \lambda_1)(a_2 + \lambda_2)]} + \tfrac{1}{2}.
\end{aligned}$$

For the sake of brevity we will now put

$$\sqrt{(a_2 + \lambda_1)} = p, \quad \sqrt{(a_2 + \lambda_2)} = q.$$

Then it is plain Ω may be written

$$\begin{aligned}
\Omega &= \frac{1 - \nu}{p + q} + \frac{\nu}{p - q} - 2\nu pq \\
&= \frac{1 - \nu}{p + q} + \frac{\nu}{p - q} - \tfrac{1}{2}\nu(p + q)^2 + \tfrac{1}{2}\nu(p - q)^2.
\end{aligned}$$

We have now to deal with T. By taking the logarithms of the values of x_1^2 and x_2^2, and then differentiating, we obtain

$$2\frac{dx_1}{x_1} = \frac{d\lambda_1}{a_1+\lambda_1} + \frac{d\lambda_2}{a_1+\lambda_2},$$
$$2\frac{dx_2}{x_2} = \frac{d\lambda_1}{a_2+\lambda_1} + \frac{d\lambda_2}{a_2+\lambda_2}.$$

Whence may be derived

$$4(dx_1^2+dx_2^2) = \left[\frac{x_1^2}{(a_1+\lambda_1)^2} + \frac{x_2^2}{(a_2+\lambda_1)^2}\right]d\lambda_1^2 + \left[\frac{x_1^2}{(a_1+\lambda_2)^2} + \frac{x_2^2}{(a_2+\lambda_2)^2}\right]d\lambda_2^2$$
$$+ 2\left[\frac{x_1^2}{(a_1+\lambda_1)(a_1+\lambda_2)} + \frac{x_2^2}{(a_2+\lambda_1)(a_2+\lambda_2)}\right]d\lambda_1 d\lambda_2.$$

On substituting in the factor of $d\lambda_1\, d\lambda_2$ the values of x_1^2 and x_2^2 it vanishes, and the expression takes the form

$$4(dx_1^2+dx_2^2) = \frac{\lambda_1-\lambda_2}{(a_1+\lambda_1)(a_2+\lambda_1)}d\lambda_1^2 + \frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)}d\lambda_2^2.$$

Or, in terms of p and q, we have

$$du^2+ds^2 = \frac{p^2-q^2}{p^2-\frac{1}{4}}dp^2 + \frac{q^2-p^2}{q^2-\frac{1}{4}}dq^2.$$

In like manner we get

$$uds - sdu = (p+q)\left[\sqrt{\left(\frac{\frac{1}{4}-q^2}{p^2-\frac{1}{4}}\right)}dp - \sqrt{\left(\frac{p^2-\frac{1}{4}}{\frac{1}{4}-q^2}\right)}dq\right].$$

The former expression for T was

$$T = \tfrac{1}{2}(1-e'\cos\epsilon')\frac{du^2+ds^2}{d\epsilon'^2} - \sqrt{(1-e'^2)}\,\frac{uds-sdu}{d\epsilon'} + \tfrac{1}{2}(1+e'\cos\epsilon')(u^2+s^2)$$
$$+ e'\sin\epsilon'\left(u\frac{du}{d\epsilon'} + s\frac{ds}{d\epsilon'}\right);$$

hence, if we abbreviate by putting

$$\sqrt{\left(\frac{\frac{1}{4}-q^2}{p^2-\frac{1}{4}}\right)} = a,$$
$$T = \tfrac{1}{2}(1-e'\cos\epsilon')\left[(1+a^2)\frac{dp^2}{d\epsilon'^2} + \left(1+\frac{1}{a^2}\right)\frac{dq^2}{d\epsilon'^2}\right]$$
$$-\sqrt{(1-e'^2)}(p+q)\left[a\frac{dp}{d\epsilon'} - \frac{1}{a}\frac{dq}{d\epsilon'}\right] + \tfrac{1}{2}(1+e'\cos\epsilon')(p+q)^2.$$
$$+\tfrac{1}{2}e'\sin\epsilon'\frac{d(p+q)^2}{d\epsilon'}.$$

T and Ω are somewhat simplified if we adopt variables ρ and σ, such that

$$p+q=\rho,\quad p-q=\sigma.$$

Also, for the sake of brevity, put

$$\tfrac{1}{2}\left(a+\frac{1}{a}\right)=h,\quad \tfrac{1}{2}\left(a-\frac{1}{a}\right)=k.$$

Then we have

$$
\begin{aligned}
T = {} & \tfrac{1}{2}(1 - e' \cos \epsilon')\left[h^2 \left(\frac{d\rho^2}{d\epsilon'^2} + \frac{d\sigma^2}{d\epsilon'^2}\right) - 2hk \frac{d\rho}{d\epsilon'}\frac{d\sigma}{d\epsilon'}\right] \\
& - \sqrt{(1 - e'^2)}\,\rho \left(k\frac{d\rho}{d\epsilon'} + h\frac{d\sigma}{d\epsilon'}\right) + \tfrac{1}{2}(1 + e'\cos\epsilon')\rho^2 + e'\sin\epsilon' . \rho \frac{d\rho}{d\epsilon'} \\
\Omega = {} & \frac{1-\nu}{\rho} + \frac{\nu}{\sigma} - \tfrac{1}{2}\nu\rho^2 + \tfrac{1}{2}\nu\sigma^2.
\end{aligned}
$$

By this transformation Ω is considerably simplified; but, as more than offsetting this, T is rendered complex. As the expression for a in terms of these variables is

$$a = \sqrt{\left[\frac{1 - (\rho - \sigma)^2}{(\rho + \sigma)^2 - 1}\right]},$$

it will be perceived that h and k are trigonometrical functions of the angles of the triangle whose sides are 1, ρ, and σ, which might have been anticipated from geometrical considerations. Thus it appears no advantage would result from the employment of elliptic co-ordinates.

Returning, therefore, to the quasi-rectangular co-ordinates u and s, it seems some advantage would be gained if we adopt a new system of co-ordinates, u and s, such that the new system is expressed, in terms of the old, as follows:—

$$u = u + s\sqrt{(-1)}, \quad s = u - s\sqrt{(-1)}.$$

We can also adopt the trigonometrical exponential corresponding to the arc ϵ' as the independent variable. Calling this $\zeta = e^{\epsilon'\sqrt{(-1)}}$, an operator D is adopted, equivalent to $\zeta \dfrac{d}{d\zeta}$, so that $D . \zeta^i = i\zeta^i$.

In terms of the new variables, Ω has the expression

$$\Omega = \frac{1-\nu}{\sqrt{(us)}} + \frac{\nu}{\sqrt{[(u-1)(s-1)]}} - \tfrac{1}{2}\nu(u+s).$$

And the differential equations are

$$
\begin{aligned}
& D\{[1 - \tfrac{1}{2}e'(\zeta + \zeta^{-1})]\,Du\} + 2\sqrt{(1 - e'^2)}\,Du + \left[1 - \tfrac{1}{4}\frac{e'^2(\zeta + \zeta^{-1})^2}{1 - \tfrac{1}{2}e'(\zeta + \zeta^{-1})}\right]u = -2\frac{\partial\Omega}{\partial s}, \\
& D\{[1 - \tfrac{1}{2}e'(\zeta + \zeta^{-1})]\,Ds\} - 2\sqrt{(1 - e'^2)}\,Ds + \left[1 - \tfrac{1}{4}\frac{e'^2(\zeta + \zeta^{-1})^2}{1 - \tfrac{1}{2}e'(\zeta + \zeta^{-1})}\right]s = -2\frac{\partial\Omega}{\partial u}.
\end{aligned}
$$

Only one of these equations need be actually employed, as either can be obtained from the other by changing the sign of $\sqrt{(-1)}$. We have

$$
\begin{aligned}
-2\frac{\partial\Omega}{\partial s} &= \frac{1-\nu}{\sqrt{u}.\sqrt{s^3}} + \frac{\nu}{\sqrt{(u-1)}.\sqrt{(s-1)^3}} + \nu, \\
-2\frac{\partial\Omega}{\partial u} &= \frac{1-\nu}{\sqrt{u^3}.\sqrt{s}} + \frac{\nu}{\sqrt{(u-1)^3}.\sqrt{(s-1)}} + \nu.
\end{aligned}
$$

For the purpose of integrating these equations, we may adopt the method of indeterminate coefficients; and we may employ, as proper to represent the values of u and s, the infinite series

$$u = \Sigma \cdot \mathfrak{a}_{i,j,k}\, \zeta^{ic + jc' + k},$$
$$s = \Sigma \cdot \mathfrak{a}_{i,j,k}\, \zeta^{-ic - jc' - k}.$$

Here i, j, and k denote positive or negative integers, zero included; and the summation must be extended so as to include all values for i, j, or k from $-\infty$ to $+\infty$. The $\mathfrak{a}$ and c, c' are constants and functions of the four quantities e', ν, a and e; a and e being two of the four arbitrary constants introduced by integration. The two remaining arbitrary constants serve only to complete the two elementary arguments which belong to the attracted planet, and, in this method of integration, they can pass unnoticed.

If we suppose that the orbit of the attracting planet is circular, the differential equations reduce to the very simple form

$$(D+1)^2 u = -2\frac{\partial \Omega}{\partial s},$$
$$(D-1)^2 s = -2\frac{\partial \Omega}{\partial u}.$$

And, in this case, an integral can be found. For multiplying the first by Ds, and the second by Du, the sum of the equations, thus multiplied, is an exact derivative. Integrating, we get

$$DuDs + us + 2\Omega = 2C,$$

C being the arbitrary constant.

This integral equation may be combined with the differential equations in such a way that one of the terms, regarded as the most difficult of expression in a developed form, may be eliminated. For example, if this is taken to be the term $\dfrac{\nu}{\sqrt{[(u-1)(s-1)]}}$ of Ω, the equations serving to determine the $\mathfrak{a}$ may be taken to be

$$(s-1)\,D(D+2)u + \tfrac{1}{2}DuDs + (1-\nu)\left[\frac{1}{\sqrt{(us)^3}} - 1\right]u + \tfrac{3}{2}(u-\nu)(s-\nu) + C = 0,$$
$$(u-1)\,D(D-2)s + \tfrac{1}{2}DuDs + (1-\nu)\left[\frac{1}{\sqrt{(us)^3}} - 1\right]s + \tfrac{3}{2}(u-\nu)(s-\nu) + C = 0,$$

in which the constant C is not identical with the former C. One of these equations suffices, as the other is a consequence of it. The difference of

these equations is simpler than either of them, and may be of use. It is

$$D[(u-1)\,Ds-(s-1)\,Du-2\,(u-1)(s-1)]=(1-\nu)\left[\frac{1}{\sqrt{(us)^3}}-1\right](u-s).$$

In attempting to derive periodic series for the co-ordinates of Hyperion, it appears to me that it will be easier, in the first instance, to assume that Titan describes a circular orbit. And in the next place, to assume that the perturbations are periodic functions of the mean elongation of the two bodies. And, as it may very easily happen that the terms, depending on the second and higher powers of the disturbing force, may quite alter the values of the coefficients, it will be well to employ the method of mechanical quadratures. Starting Hyperion from its line of conjunction with Titan, and at right angles to this line, with an assumed velocity, trace out its path until the elongation, between the two bodies amounts to 180°. Then, if Hyperion is again moving at right angles to its radius vector, the velocity at the start has been rightly assumed. But if not, one makes another trial; and, by interpolating between the two results, a velocity is obtained which will more nearly bring about this condition. And continued repetition of these trials will enable us to discover, with all desired approximation, the velocity which fulfills this condition. When the path of Hyperion, corresponding to this velocity, has been traced out, it will be easy, by the well-known processes of mechanical quadratures, to assign the periodic series representing the co-ordinates of the satellite under the supposed conditions.

When this is done, corrections to the co-ordinates, proportional to the first power of the satellite's proper eccentricity, can be obtained by the integration of a linear differential equation. By comparison of these with observation an approximate value of this proper eccentricity will be obtained; a thing to be desired as we seem to know next to nothing about it at present. Also one will be enabled to decide whether the motion of the mean anomaly is more rapid than that of the mean longitude, as has been asserted, without sufficient reason as it seems to me.

As illustrating this point, suppose that our moon, instead of having an eccentricity about 0.055, had one about 0.001. Then the variation would be the prevailing inequality, and the moon would appear to be in perigee always about syzygies, and in apogee about quadratures. In consequence the perigee would appear to retrograde with reference to the sun as fast as the moon advances with reference to the same body. And yet the relation between the motion of the argument, denominated the mean anomaly, and the motion of the mean longitude, would be nearly the same as it is at present. But the position of the perisaturnium of Hyperion has been concluded

from its observed shortest and longest *radii vectores*. This is allowable only when the inequality, called the equation of the centre, is the overpowering one.

After the terms, proportional to the first power of the eccentricity, have been obtained, those factored by the second, third, etc., powers, can be derived by integrating differential equations of the same character.

In applying the process of mechanical quadratures to the motion of Hyperion, one will meet the difficulty of the uncertain value of the mass of Titan. But this cannot be avoided; an assumption must be made, and the results afterwards corrected by comparison with observation.

MEMOIR No. 43.

On Differential Equations with Periodic Integrals.

(Annals of Mathematics, Vol. III, pp. 145-153, 1887.)

The independent variable being conceived as time, a system of differential equations may be said to admit periodic integrals when the values of the dependent variables either exactly, or with approximate tendency, after a certain lapse of time, repeat their series of values. In the latter case the larger the lapse is made the more nearly is the repetition brought about. Strange as it may seem, this subject, except in the case of simply periodic integrals, is, at present, not completely understood. The text-books on differential equations are almost wholly engaged with the cases in which, by certain artifices, the integration can be accomplished in finite terms or reduced to quadratures. In the treatment of physical problems, however, equations of this sort are rarely met with. Far more frequently it is found that methods of approximation must be resorted to. Cauchy appears to be the author who has done most for the elucidation of this part of the subject. His memoirs are in his later *Exercises* and in the volumes of the *Comptes Rendus* for 1856 and 1857. In this article I propose to show how simply periodic integrals arise and afterwards to illustrate the general theory by treating a problem relating to the motion of a system of points.

I.

Having the independent variable t, and the two dependent variables x and x_1, let us suppose the latter satisfy the equations.

$$\frac{dx}{dt} = x_1, \quad \frac{dx_1}{dt} = \mathrm{f}(x).$$

A cross multiplication between the members of these equations gives

$$x_1 \frac{dx_1}{dt} = \mathrm{f}(x) \frac{dx}{dt}.$$

The integral of this is, C being the arbitrary constant,

$$x_1^2 = 2 \int \mathrm{f}(x)\, dx + C.$$

The values of x and x_1 being known for a given value of t, we readily find the value of C proper to the special case we treat. By substituting the value of x_1 derived from this equation in the first of the differential equations we get

$$\frac{dx}{dt} = \sqrt{\left[2\int f(x)\,dx + C\right]}.$$

The expression under the radical sign is a function of x; calling it X, let us consider the equation $X = 0$. Since real values of x are supposed to correspond to all values of t, X can never be negative; and from the way the constant C was determined, it is plain that, for the given value of t, X is positive. Then in $X = 0$, let x be supposed to increase until a value $x = b$ is reached for which $X = 0$, that is to say a real root of this equation. Similarly let x diminish from the same point until a value $x = c$ is reached for which again $X = 0$, that is a second real root. Then, X being positive for all values of x which lie between c and b, if the latter are non-multiple roots, X is negative for values of x which lie just outside these limits. Thus x must necessarily remain within the limits c and b. Also, in its motion, it always attains them; for suppose x is augmenting, then the radical, which forms the value of dx/dt, must be taken positively, and, from the law of continuity, must continue to be so taken until it becomes zero, that is until x arrives at the value b. But dx/dt cannot be positive beyond this point, for x cannot surpass b. Hence, after this, the radical must receive the the negative sign, and, consequently, x begins to diminish. Again, from the law of continuity, this diminution is kept up until x has arrived at the value c. At this point the diminution must change into an augmentation, for x cannot fall below c. Thus the movement of x is a continuous swinging back and forth between the limits c and b.

We can put $$X = \frac{(b-x)(x-c)}{R^2},$$

R being a function of x which remains constantly positive and finite for all values of x between c and b. We can then write

$$\frac{dt}{dx} = \frac{R}{\sqrt{[(b-x)(x-c)]}}.$$

A new variable u can now be advantageously introduced in place of x. Let

$$x = a(1 - e\cos u),$$

where $a = \frac{1}{2}(b + c)$, and $e = (b - c)/(b + c)$; and u is equivalent to an

integral number of circumferences when $x = c$, and augments by half a circumference when x, next following, attains the value b. Thus u, like t, augments continuously. We have

$$b - x = ae(1 + \cos u), \quad x - c = ae(1 - \cos u),$$
$$\sqrt{[(b - x)(x - c)]} = ae \sin u,$$
$$dx = ae \sin u \, du.$$

Therefore

$$dt = R du.$$

As R is a one-valued function of x or of $a(1 - e \cos u)$, it can be expanded in the following periodic series

$$R = \frac{1}{n}[1 + a_1 \cos u + 2a_2 \cos 2u + 3a_3 \cos 3u + \ldots],$$

n, a_1, a_2, etc., being constants, the first having the value

$$\frac{1}{n} = \frac{1}{\pi}\int_0^\pi R du.$$

Then c being an arbitrary constant,

$$n(t + c) = u + a_1 \sin u + a_2 \sin 2u + a_3 \sin 3u + \ldots.$$

This series serves for determining t when x or u is given; but, more frequently it is x or u which is required in terms of t. It is necessary, then, to invert the series. The coefficients of the inverted series are most readily found by means of definite integrals. Let us suppose that it is required to find the periodic series, in terms of t, for a function of x and x_1 which we will denote by U. This function we assume to be always finite and continuous. The base of hyperbolic logarithms being ε, let us put

$$\zeta = n(t + c), \quad z = \varepsilon^{\zeta\sqrt{-1}}, \quad s = \varepsilon^{u\sqrt{-1}},$$

and, for brevity,

$$2S = a_1(s - s^{-1}) + a_2(s^2 - s^{-2}) + a_3(s^3 - s^{-3}) + \ldots.$$

The equation connecting z and s is

$$z = s\varepsilon^{S}.$$

We can suppose that

$$U = \sum_{i=-\infty}^{i=+\infty} A_i z^i.$$

Then

$$A_i = \frac{1}{2\pi}\int_0^{2\pi} U z^{-i} d\zeta = \frac{1}{2\pi}\int_0^{2\pi} U s^{-i} \varepsilon^{-iS} nR du.$$

U being $= F(x, x_1)$, we have

$$U = F\left\{a(1 - e\cos u),\ \frac{ae\sin u}{R}\right\}$$
$$= F\left\{a[1 - \tfrac{1}{2}e(s + s^{-1})],\ \frac{ae(s - s^{-1})}{2R\sqrt{(-1)}}\right\}.$$

Supposing that U is reduced to x, it is plain that the coefficient of z^i, in the development of x in powers of z, is the same as the coefficient of s^i in the development of

$$a[1 - \tfrac{1}{2}e(s + s^{-1})]\left[1 + s\frac{\partial S}{\partial s}\right]\varepsilon^{-iu}$$

in powers of s.

By adopting the Besselian functions $J_\lambda^{(i)}$, we have

$$\varepsilon^{-\frac{1}{2}ia_1(s - s^{-1})} = \sum_{j=-\infty}^{j=+\infty} J^{(j)}_{-\frac{1}{2}ia_1} s^j,\quad \varepsilon^{-\frac{1}{2}ia_2(s^2 - s^{-2})} = \sum_{j=-\infty}^{j=+\infty} J^{(j)}_{-\frac{1}{2}ia_2} s^{2j},\ \text{etc.};$$

and the expression, given above, can be written

$$a\left\{\begin{array}{l} 1 - \tfrac{1}{2}a_1 e \\ \quad + \left(\dfrac{a_1 - e}{2} - \dfrac{2}{2.2}a_1 e\right)(s + s^{-1}) \\ \quad + \left(\tfrac{2}{2}a_2 - \dfrac{a_1 e}{2.2} - \dfrac{3}{2.2}a_3 e\right)(s^2 + s^{-2}) \\ \quad + \left(\tfrac{3}{2}a_3 - \dfrac{2}{2.2}a_2 e - \dfrac{4}{2.2}a_4 e\right)(s^3 + s^{-3}) \\ \quad + \ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$
$$\times (\Sigma_j . J_{\lambda_1}^{(j)} s^j).(\Sigma_j . J_{\lambda_2}^{(j)} s^{2j}).\Sigma_j . J_{\lambda_3}^{(j)} s^{3j}) \ldots,$$

where $\lambda_j = -\tfrac{1}{2}i\alpha_j$.

However, unless the coefficients $\alpha_1, \alpha_2, \alpha_3, \ldots$ decrease rapidly, this will not be a practical method of developing x in a periodic series. Generally it will be shorter to employ mechanical quadratures in obtaining the value of the definite integral. Let us suppose that

$$x = \tfrac{1}{2}\beta_0 + \beta_1\cos\zeta + \beta_2\cos 2\zeta + \beta_3\cos 3\zeta + \ldots.$$

Then

$$\beta_i = \frac{2}{\pi}\int_0^\pi x\cos i\zeta\, d\zeta$$
$$= \frac{2}{\pi}\int_0^\pi a(1 - e\cos u)[1 + a_1\cos u + 2a_2\cos 2u + 3a_3\cos 3u + \ldots]\cos i\zeta\, du,$$

where, to obtain the value of ζ corresponding to a given value of u, we employ the equation

$$\zeta = u + a_1 \sin u + a_2 \sin 2u + \dots .$$

It will be seen that this method is applicable to a much wider range of questions than the motion of planets in elliptic orbits. And the superiority of the method of definite integrals over Lagrange's Theorem for the inversion of the series is quite manifest.

II.

In order to illustrate the preceding general theory, let us treat the problem of n material points moving about a centre under the action of central forces admitting a potential which is a function of the sum of the squares of the *radii vectores*. Each point will then move in a fixed plane and its *radius vector* will describe equal areas in equal times. Thus all will be virtually known in reference to these motions, provided we are able to express the *radii vectores* as functions of the time.

Let the *radii* be denoted as $r_1, r_2, \dots r_n$, and the orbit longitudes, measured each from any point in its plane, as $\lambda_1, \lambda_2, \dots \lambda_n$. For brevity, put

$$\rho^2 = r_1^2 + r_2^2 + \dots + r_n^2.$$

Then, if the potential is represented by $f(\rho)$, we shall have the two equations, representing generally all the equations of the problem,

$$\frac{d^2 r_i}{dt^2} - r_i \frac{d\lambda_i^2}{dt^2} = f'(\rho) \frac{r_i}{\rho},$$

$$\frac{d\lambda_i}{dt} = \frac{h_i}{r_i^2},$$

h_i being the constant of areolar velocity. Consequently if we put

$$\Omega = f(\rho) - \tfrac{1}{2}\Sigma \frac{h_i^2}{r_i^2},$$

the general form of the differential equations determining the *radii vectores* will be

$$\frac{d^2 r_i}{dt^2} = \frac{\partial \Omega}{\partial r_i}.$$

They have the integral, corresponding to that of living forces,

$$\Sigma \frac{dr_i^2}{dt^2} = 2(\Omega + C),$$

C being an arbitrary constant. Also we may derive

$$\Sigma r_i \frac{d^2 r_i}{dt^2} = \Sigma r_i \frac{\partial \Omega}{\partial r_i}.$$

By adding the last two equations,

$$\frac{d}{dt}\left(\rho \frac{d\rho}{dt}\right) = 2f(\rho) + \rho f'(\rho) + 2C,$$

an equation involving only the dependent variable ρ. Multiplying it by the factor $2\rho \frac{d\rho}{dt}$, and integrating, we get, A being an arbitrary constant,

$$\rho^2 \frac{d\rho^2}{dt^2} = 2\rho^2 [f(\rho) + C] - A^2.$$

Whence $$t + c = \int \frac{\rho d\rho}{\sqrt{\{2\rho^2 [f(\rho) + C] - A^2\}}}.$$

Inverting this we shall have ρ as a function of t.

By dividing the penultimate equation by ρ^2 and differentiating, we get

$$\frac{d^2\rho}{dt^2} = \frac{f'(\rho)}{\rho}\rho + \frac{A^2}{\rho^3}.$$

The general equation determining r_i is

$$\frac{d^2 r_i}{dt^2} = \frac{f'(\rho)}{\rho} r_i + \frac{h_i^2}{r_i^3}.$$

As ρ is now a known function of t, r_i is the only unknown in it, and consequently, the equation by itself suffices for determining it. To put the equation in a form suitable for integration, let us eliminate $f'(\rho)$ between the last two equations. We get

$$\frac{d(\rho dr_i - r_i d\rho)}{dt^2} = \frac{h_i^2}{r_i^3}\rho - \frac{A^2}{\rho^3} r_i,$$

or $$\rho^2 \frac{d}{dt}\left[\rho^2 \frac{d}{dt}\left(\frac{r_i}{\rho}\right)\right] = \left[\frac{h_i^2 \rho^4}{r_i^4} - A^2\right]\frac{r_i}{\rho}.$$

To simplify this, we will adopt an auxiliary variable ψ, such that

$$d\psi = \frac{A}{\rho^2} dt = \frac{A d\rho}{\rho \sqrt{\{2\rho^2 [f(\rho) + C] - A^2\}}}.$$

Then $$\frac{d^2\left(\frac{r_i}{\rho}\right)}{d\psi^2} = \left[\frac{h_i^2}{A^2}\frac{\rho^4}{r_i^4} - 1\right]\frac{r_i}{\rho}.$$

Whence, by integration, we derive

$$\left\{\frac{d\left(\frac{r_i}{\rho}\right)}{d\psi}\right\}^2 = 2\mathrm{a}_i - \frac{r_i^2}{\rho^2} - \frac{h_i^2\rho^2}{A^2 r_i^2},$$

a_i being the arbitrary constant. By putting

$$\frac{r_i}{\rho} = \sqrt{(u_i)},$$

we get

$$d\psi = \frac{du_i}{2\sqrt{\left[2\mathrm{a}_i u_i - u_i^2 - \frac{h_i^2}{A^2}\right]}}.$$

For convenience, adopting a new constant e_i, in place of h_i, such that $h_i^2/A^2 = \mathrm{a}_i^2(1 - e_i^2)$ the quantity under the radical sign becomes

$$[\mathrm{a}_i(1 + e_i) - u_i][u_i - \mathrm{a}_i(1 - e_i)].$$

Thus, putting $u_i = \mathrm{a}_i(1 - e_i \cos \varepsilon_i)$, ε_i being a new variable, we get $d\psi = \frac{1}{2}d\varepsilon_i$, and thus $\varepsilon_i = 2\psi + \alpha_i$, α_i being a constant. Thus we have, in fine,

$$\frac{r_i}{\rho} = \sqrt{\{\mathrm{a}_i[1 - e_i \cos(2\psi + 2\alpha_i)]\}}.$$

As we have

$$\Sigma \frac{r_i^2}{\rho^2} = 1,$$

the constants a_i, e_i, and α_i satisfy the relations

$$\Sigma \mathrm{a}_i = 1, \quad \Sigma \mathrm{a}_i e_i \cos 2\alpha_i = 0, \quad \Sigma \mathrm{a}_i e_i \sin 2\alpha_i = 0.$$

We thus have $2n$ independent arbitrary constants introduced by integration; the number there should be.

In order to find an expression for the longitudes, we take the general equation

$$d\lambda_i = \frac{h_i dt}{\mathrm{a}_i \rho^2 [1 - e_i \cos 2(\psi + \alpha_i)]}$$
$$= \frac{\sqrt{(1 - e_i^2)}\, d\psi}{1 - e_i \cos 2(\psi + \alpha_i)}.$$

The integral of which gives

$$\tan(\lambda_i + \beta_i) = \sqrt{\left(\frac{1 + e_i}{1 - e_i}\right)}.\tan(\psi + \alpha_i),$$

β_i being the arbitrary constant.

To simplify the equations which give $t + c$ and ψ, we suppose that $a(1 + e)$ is the maximum value of ρ, and $a(1 - e)$ its minimum value. Then

we can adopt a variable ε such that

$$\rho = a(1 - e\cos\varepsilon).$$

Thus $d\rho = ae\sin\varepsilon d\varepsilon$, and we may put

$$2\rho^2[f(\rho) + C] - A^2 = R^2a^2e^2\sin^2\varepsilon,$$

where R remains constantly positive throughout the motion of ρ. Then

$$t + c = \int \frac{\rho}{R}d\varepsilon,$$

$$\psi = \int \frac{A}{\rho R}d\varepsilon.$$

R, being a function of ρ, is also one of a $(1 - e\cos\varepsilon)$, and thus is capable of being expanded in a converging series of terms, each consisting of a constant multiplied by the cosine of a multiple of ε. Also ρ/R and $A/\rho R$ can be expanded in similar series. Then the period T, in which ρ goes through the round of its values, is given by the definite integral

$$T = \int_0^{2\pi} \frac{a(1 - e\cos\varepsilon)}{R}d\varepsilon,$$

and the augmentation of the variable ψ, in the same time, will be equivalent to the definite integral

$$\int_0^{2\pi} \frac{Ad\varepsilon}{a(1 - e\cos\varepsilon)R}.$$

If the value of the latter is 2π, ψ will augment by a circumference while ρ goes through its period. This is the case when $f(\rho) = \mu/\rho$; but, in general, this condition is not fulfilled.

Provided that A^2 is a positive quantity, it is plain that, after ψ has gone through its period, the longitudes and latitudes, whether as seen from the centre or from any of the points, all return to the same values. The same thing is true of the ratios of the *radii vectores*. Thus the movement of the system may be conceived as taking place under the operation of two distinct causes. The first producing a revolution of all the points about the centre in closed curves and in the same time, while the second, having a different period, changes the scale of representation of the system in space.

In the preceding treatment we have supposed that A^2 is a positive quantity. When this is not the case, some modifications must be made. Let us

suppose first that $A = 0$. Then we have

$$t + c = \int \frac{d\rho}{\sqrt{\{2[\mathfrak{f}(\rho) + C]\}}},$$

and we may assume

$$\psi = \int \frac{d\rho}{\rho^2 \sqrt{\{2[\mathfrak{f}(\rho) + C]\}}}.$$

Then

$$\left\{ \frac{d\left(\frac{r_i}{\rho}\right)}{d\psi} \right\}^2 = \mathfrak{a}_i - h_i^2 \frac{\rho^2}{r_i^2},$$

$$\psi + a_i = \sqrt{\left(\mathfrak{a}_i \frac{r_i^2}{\rho^2} - h_i^2\right)},$$

$$\frac{r_i}{\rho} = \sqrt{\left(\frac{(\psi + a_i)^2 + h_i^2}{\mathfrak{a}_i}\right)}.$$

Also

$$d\lambda_i = \frac{h_i dt}{r_i^2} = h_i \frac{\rho^2}{r_i^2} d\psi$$

$$= \frac{h_i \mathfrak{a}_i d\psi}{(\psi + a_i)^2 + h_i^2},$$

$$\tan(\lambda_i + \beta_i) = \frac{\mathfrak{a}_i}{h} (\psi + a_i).$$

In the second place, let A^2 be negative. Here it is only necessary in some places to accomplish the integrations by the aid of hyperbolic cosines instead of circular.

The differential equations of this problem, in the case where the *radii* are supposed to describe no areas, were first integrated by Binet.* But the addition, to the forces, of the terms arising from centrifugal action, much enhances the interest of the problem.

* See Liouville, *Journal de Mathématiques*, First Ser., Tome II, p. 457.

MEMOIR No. 44.

On the Interior Constitution of the Earth as Respects Density.

(Annals of Mathematics, Vol. IV, pp. 19–29, 1888.)

Nearly all the matter accessible to us is found to be porous. Thus the application of pressure to it tends to reduce the amount of porosity and, in consequence, augments the density of the mass. Moreover, the greater the pressure the greater is the increment of density. A familiar instance of this is the case of atmospheric air or a gas in which, provided the temperature remains constant, the density varies directly as the pressure.

It is natural to think that the matter of which the earth is composed is not excepted from this law. At small depths, it is true, the rigidity of the earth's mass interferes with its exerting any pressure, as the existence of caves shows. But at great depths where the weight of the superincumbent mass becomes very great, it is extremely probable the molecular force of cohesion gives way in a manner which allows pressure to act; which is illustrated by the behavior of ice in a glacier.

I propose to see what conclusions we are led to by adopting this relation between the density ρ and the pressure p,

$$\rho = A + Bp.$$

A and B are constants, A denoting the density at the surface, and B the rate of increase of the density per unit of pressure. In applying this formula to the atmosphere and gases, we have by Boyle's law $A = 0$. Let V denote the potential of the gravitating force of the whole mass, and let us neglect the effect of the centrifugal force arising from the rotation of the earth. Then pressure being supposed to act as though the whole mass were fluid, hydrostatics furnishes us with the equation

$$dp = \rho dV.$$

V being restricted to points on the surface or in the interior of the mass, it satisfies the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0.$$

The three equations now written may be regarded as determining the three unknowns ρ, p, and V.

By the elimination of V and p we get

$$\frac{\partial^2 \log \rho}{\partial x^2} + \frac{\partial^2 \log \rho}{\partial y^2} + \frac{\partial^2 \log \rho}{\partial z^2} + 4\pi B\rho = 0.$$

It will be seen that the constant A has disappeared from this equation. By Boyle's law in the case of gases $A = 0$; that is, the matter is capable of attenuating itself to an infinite degree, a thing very improbable. But the introduction of the constant term A, and consequent supposition of a limit to the attenuation, does not change the differential equation which ρ satisfies. This partial differential equation contains the whole theory of gases under a uniform temperature contained in vessels of any figure, and acted on by any gravitating forces; also the theory of atmospheres surrounding solid nuclei of density as heterogeneous as we please, and of any figure. The truth of the equation is not at all invalidated by any discontinuity in ρ or B; these quantities may change the law of their values as often as the problem demands.

The very simple integral of this equation in the case of the earth's atmosphere, when the attraction of the atmosphere on itself is neglected, is well known. It is our object here to examine the special solutions of this equation which are defined by the equation,

$$\rho = \text{function}\left[\sqrt{(x^2 + y^2 + z^2)}\right].$$

In this case, making $r = \sqrt{(x^2 + y^2 + z^2)}$, the partial differential equation is reduced to an ordinary one and becomes

$$\frac{d \, . \, r^2 \dfrac{d \, . \log \rho}{dr}}{dr} + 4\pi B r^2 \rho = 0,$$

or, as it may be written,

$$\frac{d^2 (r \log \rho)}{dr^2} + 4\pi B r \rho = 0.$$

To simplify this, let us put

$$s = 4\pi B r^2 \rho.$$

Then s being made the dependent variable, we have

$$\frac{d \, . \, r^2 \dfrac{d \, . \log s}{dr}}{dr} + s - 2 = 0.$$

And if $\log r = v$, it becomes

$$\frac{d^2 \log s}{dv^2} + \frac{d.\log s}{dv} + s - 2 = 0.$$

Futhermore, if $\frac{d.\log s}{dv} = u$, this differential equation of the first order between u and s is obtained

$$\frac{du}{ds} = \frac{2 - (u + s)}{us}.$$

This being integrated, and u obtained in terms of s, or s in terms of u, r is given by the equation

$$r = K\epsilon^{\int \frac{ds}{us}},$$

or by the equation

$$r = K\epsilon^{\int \frac{du}{2-(u+s)}},$$

in which K is an arbitrary constant. And, if in the first of these values of r, $4\pi Br^2\rho$ is substituted for s, the equation will be obtained which determines ρ as a function of r.

The differential equation in u and s is a particular case of the general form

$$Pdx + Qdy = 0,$$

where P and Q denote algebraical functions of x and y of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Mathematicians have been able to obtain the integral of this, in finite terms, only when the constants A, B, etc., satisfy certain equations of condition.* Unfortunately, the differential equation under consideration does not belong to any of these particular cases. Recourse must be had to series or other methods of approximation for the determination of the relation between u and s. However, the differential equation itself will furnish the properties of the family of plane curves it defines.

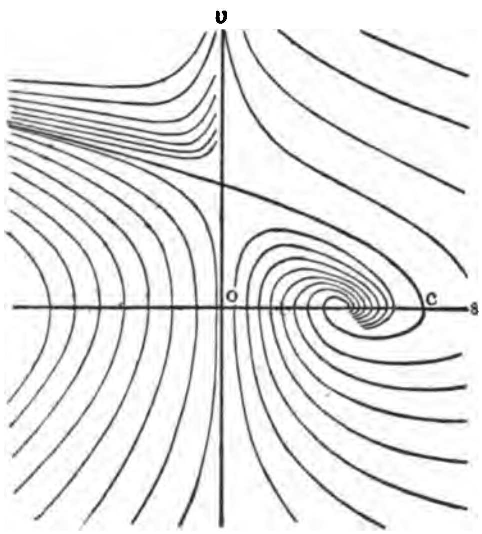
Thus u and s denoting the rectangular co-ordinates of a point in a plane, the differential equation gives immediately the means of drawing the tangent to the curve which passes through this point. Excepting at the two singular points whose co-ordinates are $u = 0$, $s = 2$ and $u = 2$, $s = 0$, for

* See Liouville, *Journal de Mathématiques*, 2e Series, Tom. III, p. 417.

which the expression of the tangent takes the indeterminate form

$$\frac{du}{ds}=\frac{0}{0},$$

the curves do not intersect each other, since there is but one value of $\frac{du}{ds}$ for given values of u and s. Since the differential equation is satisfied by the condition $s=0$, the axis of u is itself one of the system of curves, and no curve can cross it except at the point $u=2$. If, in the differential equation, we substitute $2+du$ for u, and ds for s, it is clear that only one curve passes through this point, and that its tangent here is given by the equation $du/ds=-\frac{1}{3}$. The axis of u, between the points $u=2$ and $u=\infty$, is an asymptote to the whole system of curves. The axis of s is intersected at right angles by the system of curves. Investigating what occurs at the point $s=2$ on this axis, we substitute du for u and $2+ds$ for s, and obtain for determining du/ds at this point the following quadratic



$$\left(\frac{du}{ds}\right)^2+\frac{1}{3}\frac{du}{ds}+\frac{1}{3}=0,$$

the roots of which are imaginary. Hence no curve passes through this point, and it is easy to see that the system of curves makes an infinite number of turns about it.

The tangent to any curve, at its intersection with the straight line whose equation is $u+s=2$, is parallel to the axis of s. When u and s are both very great, the tangent to the curve approximates to parallelism with the axis of s. When s is very great and u small in comparison, the differential equation becomes approximately

$$u\frac{du}{ds}=-1;$$

or integrated,

$$u^2=2(s_0-s),$$

if s_0 is the value of s when $u=0$. Hence the curves in the vicinity of the axis of s approximate to the parabola, in measure as we recede from the origin of co-ordinates.

It is very easy to draw the curves connecting all the points possessing

parallel tangents. For convenience let α denote the common value of ds/du for these points; then the differential equation furnishes

$$(u + \alpha)(s + \alpha) = \alpha(\alpha + 2).$$

Thus these curves are equilateral hyperbolas having their asymptotes parallel to the axis of co-ordinates.

Thus much in regard to the properties of the curves defined by the differential equation under consideration. But, for the special physical problem we have in view, there is no necessity to attend to the course of the curves through the whole plane. The density being supposed to increase with augmentation of pressure, B is necessarily positive, and r and ρ, from the nature of the problem, being the same; s is likewise a positive quantity. There is then need only of considering the curves on the positive side of the axis of u. Moreover, since

$$u = \frac{d \, . \log (r^2 \rho)}{d \, . \log r} = \frac{r}{\rho}\frac{d\rho}{dr} + 2,$$

and $d\rho/dr$ is always negative when the force is directed towards the centre of the mass, there is no need of attending to the curves in the portion of the plane for which $u > 2$.

Before proceeding to the special problem we have in hand, I propose to illustrate the general theory by considering the density of the earth's atmosphere. It must be remembered that, in the usual manner of treating this question, the attraction of the atmosphere on itself is neglected; here, however, it is taken into account. Boyle's law being supposed to hold exactly, we shall have

$$\rho = Bp.$$

To integrate the differential equation between u and s, it will be necessary to obtain from observation the initial values of these two variables which hold at the surface of the earth. Let us denote these by u_0 and s_0; and by a similar notation the values of all the variables at the earth's surface. The values of u_0 and s_0 result from those of certain well-known physical constants.

Let

$D =$ the density of mercury,
$h =$ the altitude of the barometer,
$g =$ the force of gravity,
$R =$ the mean density of the earth.

From an equation just given we have

$$u_0 = r_0 \left(\frac{d \, . \log \rho}{dr}\right)_0 + 2$$
$$= \frac{r_0}{p_0}\left(\frac{dp}{dr}\right)_0 + 2.$$

But we also evidently have

$$p_0 = gDh,$$
$$\left(\frac{dp}{dr}\right)_0 = -g\rho_0.$$

Substituting these values, $$u_0 = 2 - \frac{\rho_0 r_0}{Dh}.$$

Thus it is apparent that u is independent of the units assumed for the measurement of lengths and densities. In the next place

$$B = \frac{\rho_0}{p_0} = \frac{\rho_0}{gDh}.$$

But we have $$g = \frac{4\pi R r_0^3}{3} \cdot \frac{1}{r_0^2} = \tfrac{4}{3}\pi R r_0.$$

Thence we get $$s_0 = 4\pi B r_0^2 \rho_0 = \frac{3\rho_0^2 r_0}{DRh}.$$

Thus s is also independent of the just mentioned units.

Let us adopt the following values of the constants which enter into the expressions of u_0 and s_0:—

$$\begin{aligned} r_0 &= 6365419 \text{ metres}, \\ h &= 0.76 \text{ metres}, \\ \rho_0 &= 0.001293187, \\ D &= 13.596, \\ R &= 5.67. \end{aligned}$$

The value of ρ_0 is that found by Regnault* for the temperature 0° of the centigrade scale and the given altitude of the barometer; r_0 is the distance of his observatory from the centre of the earth according to Bessel's dimensions of the terrestrial spheroid; and the value of R is that determined by Baily in his repetition of the Cavendish experiment. With these data we obtain the following values of u_0 and s_0:—

$$\begin{aligned} u_0 &= -794.6425, \\ s_0 &= \quad 0.5450835. \end{aligned}$$

Having these initial values we can easily integrate the differential equation connecting u and s by mechanical quadratures or series, in the direction of s diminishing until s becomes so small as to be of no account. The corresponding values of r and ρ could then be found as we have already explained. However, the differences between the numerical values obtained

**Mémoires de l'Académie des Sciences de Paris*, Tom. XXI.

by this method and those resulting from neglecting the action of the atmosphere on itself would be insensible.

We pass now to the problem of the mass of the earth. Let us here denote the values of the variables which hold at the centre by the subscript ($_0$). If the density at the centre be finite we must have $s_0 = 0$; and the differential equation

$$\frac{ds}{du} = \frac{us}{2-(u+s)}$$

shows that $u_0 = 2$, else s would be 0 for all values of u. Hence the curve we have to consider, in this case, is the single one which passes through the singular point $u = 2$, $s = 0$.

The mass included in the sphere whose radius is r, is

$$\begin{aligned} M &= \frac{1}{B}\int_0^r s dr \\ &= -\frac{1}{B} r^2 \frac{d.\log\rho}{dr} \\ &= \frac{1}{B} r(2-u). \end{aligned}$$

Hence, denoting the values of the variables at the earth's surface by the subscript ($_1$), and R denoting, as before, the mean density of the earth, we shall have

$$\frac{4\pi}{3} Rr_1^{\,3} = \frac{r_1(2-u_1)}{B}.$$

Whence we derive $$B = \frac{3(2-u_1)}{4\pi Rr_1^{\,2}},$$

and $$s_1 = 3(2-u_1)\frac{\rho_1}{R}.$$

Then if we draw in the plane the right line whose equation is

$$s = 3\frac{\rho_1}{R}(2-u),$$

the co-ordinates of its intersection with the curve defined by the differential equation and passing through the singular point $u = 2$, $s = 0$, will be the values of u_1 and s_1. This right line passes through the point $u = 2$, $s = 0$, and it is readily ascertained from the differential equation that upon this curve u constantly diminishes as s augments until it becomes 0. The lines can therefore intersect on the positive side of the axis of s only when

$$6\frac{\rho_1}{R} > OC,$$

where OC is the distance from the origin of the point where the mentioned curve crosses the axis of s.

In order to illustrate the general theory by an application, I have computed by mechanical quadratures the values of the variable s and the function necessary for obtaining r. For this purpose it will be well to substitute for the independent variable u the variable $z = 2 - u$. The results obtained are given in the following table at intervals of 0.1 in z:—

z	s	s/z	$\int\frac{ds}{z-s}$	$\log r$	$\log s/r^3$
0.0	0.000	3.000	$-\infty$	$-\infty$	0.4771
0.1	0.294	2.940	-1.1360	9.5065	0.4553
0.2	0.576	2.879	-0.7737	9.6640	0.4323
0.3	0.846	2.818	-0.5546	9.7592	0.4088
0.4	1.103	2.757	-0.3938	9.8290	0.3845
0.5	1.348	2.695	-0.2646	9.8851	0.3594
0.6	1.580	2.633	-0.1551	9.9326	0.3333
0.7	1.799	2.570	-0.0589	9.9744	0.3061
0.8	2.005	2.507	$+0.0279$	0.0121	0.2780
0.9	2.198	2.442	$+0.1078$	0.0468	0.2485
1.0	2.378	2.378	$+0.1825$	0.0792	0.2176
1.1	2.543	2.312	$+0.2533$	0.1100	0.1854
1.2	2.695	2.246	$+0.3213$	0.1396	0.1514
1.3	2.832	2.178	$+0.3874$	0.1682	0.1155
1.4	2.953	2.110	$+0.4522$	0.1964	0.0776
1.5	3.060	2.040	$+0.5163$	0.2242	0.0372
1.6	3.149	1.968	$+0.5806$	0.2522	9.9939
1.7	3.222	1.895	$+0.6457$	0.2804	9.9473
1.8	3.276	1.820	$+0.7123$	0.3094	9.8966
1.9	3.310	1.742	$+0.7816$	0.3394	9.8414
2.0	3.322	1.661	$+0.8547$	0.3712	9.7791
2.1	3.309	1.576	$+0.9336$	0.4055	9.7088
2.2	3.265	1.484	$+1.0215$	0.4436	9.6266
2.3	3.182	1.384	$+1.1239$	0.4881	9.5365

Let us suppose that the surface density of the earth $\rho_1 = 2.7$ and the mean density $R = 5.67$. Then at the surface of the earth the value of s/z must be

$$\frac{s_1}{z_1} = 3\frac{\rho_1}{R} = 1.4286.$$

By interpolating in the table it is found that this value corresponds to the following values of the principal variables:—

$$z = 2.257,$$
$$s = 3.224,$$
$$\log r = 0.4681,$$
$$\log\frac{s}{r^3} = 9.5722.$$

Now the last two quantities are the logarithms of the surface values of the radius and the density measured in such units as in every case will give the simplest values to the arbitrary constants. But let us take the radius at the

surface as the linear unit, and represent the surface density as 2.7. Then to reduce the numbers so as to correspond to these units, it is evident we must add 9.5319 to the logarithms in the column of $\log r$, and 0.8592 to the logarithms in the column of $\log s/r^3$. Thus are obtained the following corresponding values of r and ρ:—

r	ρ	r	ρ
0.000	21.69	0.469	10.25
0.109	20.63	0.501	9.43
0.157	19.57	0.535	8.65
0.195	18.54	0.570	7.88
0.230	17.53	0.608	7.13
0.261	16.54	0.649	6.40
0.291	15.58	0.694	5.70
0.321	14.63	0.743	5.02
0.350	13.72	0.800	4.35
0.379	12.81	0.866	3.70
0.408	11.93	0.945	3.06
0.438	11.08	1.000	2.70

It will be noticed that the density at the centre is almost double of that given by Laplace's formula; and it seems that this supposition as to the law of density will not fit the phenomena as well as the latter.

The limit beneath which the ratio ρ_1/R cannot be reduced without the problem failing to have a solution, is of interest. If the curve employed for the solution of this problem is prolonged until its tangent passes through the singular point on the axis of u, which it plainly must do before the curve crosses the axis of s a second time, this tangent affords the limit sought for the ratio $3\rho_1/R$. The tangents of the curves, at the points of the plane whose co-ordinates satisfy the equation

$$\frac{2-(u+s)}{us}=\frac{u-2}{s},$$

pass through the mentioned singular point. This equation in a simpler form is

$$s=(1+u)(2-u),$$

which consequently represents a parabola passing through both singular points, and having its axis parallel to that of s. By the employment of mechanical quadratures, the following additional points of the curve have been obtained:—

s	z	s	z
3.0	2.420	2.3	2.499
2.9	2.458	2.2	2.478
2.8	2.486	2.1	2.446
2.7	2.505	2.0	2.403
2.6	2.515	1.9	2.345
2.5	2.518	1.8	2.264
2.4	2.513	1.75	2.204

From these it is evident the point $u = -0.2$, $s = 1.76$ which lies on the just-mentioned parabola is also very nearly on the employed curve. Hence if ρ_1/R is less than a fraction which is approximately $\frac{4}{15}$, there is no solution.

The number of solutions in any particular case is deserving of notice. The integral

$$\int \frac{dz}{s - z}$$

is proportional to the value of $\log r$. It does not become infinite until the curve has made an infinite number of turns about the singular point on the axis of s. This may be shown by a transformation of variables. Let us adopt polar co-ordinates, the singular point being the pole, and thus put

$$s = \mathrm{w} \cos \theta + 2\,,$$
$$z = \mathrm{w} \sin \theta + 2\,,$$

The differential equation then becomes

$$\frac{d\mathrm{w}}{\mathrm{w}} = -\frac{\mathrm{w} \sin \theta \cos^2 \theta + \sin^2 \theta + \sin \theta \cos \theta}{\mathrm{w} \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta}\, d\theta\,.$$

And we have

$$\int \frac{dz}{s - z} = \int \frac{d\theta}{\mathrm{w} \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta}\,.$$

The denominator of these expressions cannot vanish unless w exceeds 2, and it is plain that it remains positive and finite for all values of θ. Thus r becomes infinite only when θ does. Consequently there are an infinite number of solutions when $\rho_1/R = \frac{1}{8}$; and a finite number when ρ_1/R is either less or greater than this. With the value we have attributed to this fraction in the case of the earth, the course of the curve shows that there is but one solution.

MEMOIR No. 45.

The Motion of Hyperion and the Mass of Titan.

(Astronomical Journal, Vol. VIII, pp. 57–62, 1888.)

The diversity of the values assigned to the mass of Titan, the bright satellite of Saturn, has led me to look into the matter. No doubt it will seem of more importance to the practical astronomer to make close predictions of the future positions of Hyperion than merely to gratify a scientific curiosity as to the mass of Titan. But the attainment of the first end may be very much facilitated by correct knowledge as to the latter element.

I begin with certain generalities in reference to the problem of three bodies. Let us suppose that two planets or satellites are circulating about their central body in the same plane, and that their motion is of a stable character. Then, adopting the notation of Delaunay, D the mean elongation, l the mean anomaly of the one and l' that of the other, the longitudes and radii can be expressed, in a convergent manner, by infinite series of the forms

$$\begin{aligned} V \text{ or } V' &= \text{mean long.} + \Sigma A \sin (iD + jl + j'l') \\ r \text{ or } r' &= \phantom{\text{mean long.} +{}} \Sigma B \cos (iD + jl + j'l'). \end{aligned}$$

Here i, j and j' are positive or negative integers, and the coefficients A and B have, as a factor, $e^{\pm j} e'^{\pm j'}$, where the ambiguous signs are so taken that the exponents may be positive. From whatever points in the plane we suppose that the planets set out, e and e' depend on the initial velocities and their directions. Then the latter can be so adjusted that we have $e = 0$ and $e' = 0$. It will be seen that this is equivalent to making four out of the eight arbitrary constants of the problem vanish. In this case we have

$$\begin{aligned} V \text{ or } V' &= \text{mean long.} + \Sigma A \sin iD \\ r \text{ or } r' &= \phantom{\text{mean long.} +{}} \Sigma B \cos iD. \end{aligned}$$

The inequalities of the longitudes and the radii can therefore be tabulated in tables to single entry with the argument D. Differentiating the second equation we obtain

$$\frac{dr}{dt} \text{ or } \frac{dr'}{dt} = -(n - n') \, \Sigma iB \sin iD$$

which shows that, in conjunction or opposition, not only are the true longitudes equivalent to the mean, but that then the planets move perpendicularly to their radii. This does not exclude the possibility of their so moving at other points of their orbits; in the case of Hyperion this particular direction of motion occurs twice between conjunction and opposition.

The possibility of the special case of the problem of three bodies which has just been described may be still further illustrated. Let, at a certain moment, the planets be seen in conjunction from the central body. If, at this moment, the directions of their motions relative to the central body are perpendicular to their radii and in the same plane, the circumstances of their motion, before and after the mentioned conjunction, are identical but in reverse order with respect to the time. That is, if t the time is counted from the moment of conjunction, the radii will be functions of t^2; and if the longitudes of the planets are counted from the line of the conjunction they will be equivalent to functions of t^2 multiplied by t. For let us grant that the longitudes are measured in the reverse direction, and that time past is considered as future. These changes are effected by writing $-t$, $-V$ and $-V'$ for t, V and V' in the differential equations of motion. They are unaltered by this. In addition the four quantities

$$\frac{dr}{dt} = 0, \frac{dr'}{dt} = 0, \frac{dV}{dt} \text{and} \frac{dV'}{dt}$$

are the same in both cases. Thus is apparent the truth of our statement.

The planets now setting out from conjunction, one will generally have a more rapid motion in longitude than the other. Let this be the one nearer the central body, and let the motion of both be followed until the angular distance between them has reached 180°, or until they are seen in opposition at the central body. We may now consider the angles the directions of their motions at this time form with their radii. With velocities assigned at random to them at the moment of starting from conjunction, they will, most probably, reach the state of opposition with these angles somewhat different from right angles. But, provided that the ratios of the two planetary masses to that of the central body, and the ratio of the radii at the moment of conjunction are contained within certain limits, which undoubtedly leave a large field for selection of values, it will be found that we can adjust the initial velocities of the two planets in such a manner that, when they reach the state of opposition, they will again move perpendicularly to their radii.

Granting that this adjustment has been made, it is evident, from the same reasoning as before, that the circumstances of motion of the planets, before and after the moment of opposition, are identical, but in reverse order with respect to the time. It follows from this that, the motion being continued, the planets will advance from opposition to conjunction again in the same time as they took to pass from conjunction to opposition; and when they arrive there will have the same radii and the same velocities as when they last were in conjunction. Hence, in passing from one conjunction to

the next, they have gone through a complete round of all the phases of their motions relatively to each other and to their central body.

When the principle of Fourier's theorem is invoked to supply us the periodic series exhibiting the values of the co-ordinates, it is readily seen that they depend on a single argument as D which augments by a circumference during a synodic period of the two planets, and that they have the forms which have already been given.

From the observations which have been made of Hyperion it appears that it is quite approximately in the case we have described, that is to say that its radius is very nearly at a standstill when it is either in conjunction or opposition with Titan. It is true that Titan is known to have a proper eccentricity of 0.028, which must trouble to some extent this condition of motion. But it seems quite legitimate to neglect this effect in a first approximation, and it is proposed to solve the problem of the perturbations of Hyperion and the mass of Titan as if the mentioned condition were vigorously fulfilled. The problem is simplified by assuming that the mass of Hyperion is insensible, and, consequently, that Titan moves uniformly in a circular orbit.

The elements needed for the solution, and which must be furnished by observation, are four in number. Those which will be here employed are as follows:

Daily motion of Titan	$= 22°.5770090$
Average daily motion of Hyperion	$= 16°.9198837$
Constant radius of Titan	$= 176''.915$
Radius of Hyperion in opposition	$= 192''.582$

The first of these data is due to Bessel, whose elements of Titan appear to be still not antiquated. The remaining three are due to Prof. Asaph Hall, Hyperion's a being multiplied by 0.9 to produce the opposition radius. From these data we get the following deductions:

Synodic period	$= 63^d.6365612$
Half synodic period	$= 31^d.8182806$
Motion of Titan in half synodic period	$= 718°.361609$
" " Hyperion in half syn. per.	$= 538°.361609$
" " Conj. line " "	$= -1°.638391.$

Calling the angle the direction of motion makes with the radius ψ, the equation for ψ is

$$\cot. \psi = \frac{e}{\sqrt{(1-e^2)}} \sin E.$$

Supposing that Hyperion sets out from opposition as its perisaturnium with an eccentricity $= 0.1$, at conjunction, without any action from Titan,

we shall have $\psi = 90° \; 8' \; 58''.85$. But through the action of Titan this is reduced to 90°. This is a permanent effect, and may be used to discover the mass of Titan.

And, in order to get a preliminary value of this mass to be used in the more serious portion of the work, I computed the motion of the line of apsides during the half synodic period from opposition to conjunction, neglecting all but the first power of the disturbing force. The mass of Titan was put = 0.0001, Hyperion's eccentricity = 0.1 and half a day was adopted as the interval. The result is shown in the following table:

	$\sum\frac{d\omega}{dt}$	$\frac{d\omega}{dt}$		$\sum\frac{d\omega}{dt}$	$\frac{d\omega}{dt}$		$\sum\frac{d\omega}{dt}$	$\frac{d\omega}{dt}$
d	″	″	d	″	″	d	″	″
0.0	0.000		11.0	+349.682		22.0	− 61.582	
		−33.977			− 2.765			+45.620
0.5	− 33.977		11.5	346.917		22.5	15.962	
		32.451			15.420			42.294
1.0	66.428		12.0	331.497		23.0	+ 26.332	
		29.427			27.793			36.607
1.5	95.855		12.5	303.704		23.5	62.939	
		24.962			39.297			29.508
2.0	120.817		13.0	264.407		24.0	91.447	
		19.158			49.358			17.915
2.5	139.975		13.5	215.049		24.5	109.362	
		12.171			57.446			+ 4.726
3.0	152.146		14.0	157.603		25.0	114.088	
		− 4.218			63.104			−11.124
3.5	156.364		14.5	94.499		25.5	102.964	
		+ 4.419			65.972			29.741
4.0	151.945		15.0	+ 28.527		26.0	73.223	
		13.398			65.820			51.180
4.5	138.547		15.5	− 37.293		26.5	+ 22.043	
		22.332			62.622			75.448
5.0	116.215		16.0	99.915		27.0	− 53.405	
		30.797			56.464			102.533
5.5	85.418		16.5	156.379		27.5	155.938	
		38.373			47.529			132.593
6.0	47.045		17.0	203.908		28.0	288.531	
		44.650			36.373			165.886
6.5	− 2.395		17.5	240.281		28.5	454.417	
		49.260			23.687			202.750
7.0	+ 46.865		18.0	263.968		29.0	657.167	
		51.898			−10.270			243.752
7.5	98.763		18.5	274.238		29.5	900.919	
		52.342			+ 3.045			289.048
8.0	151.105		19.0	271.193		30.0	1189.967	
		50.468			15.423			338.400
8.5	201.573		19.5	255.770		30.5	1528.367	
		46.259			26.226			388.508
9.0	247.832		20.0	229.544		31.0	1916.875	
		39.809			34.936			430.527
9.5	287.641		20.5	194.608		31.5	2347.402	
		31.326			41.278			450.091
10.0	318.967		21.0	153.330		32.0	−2797.493	
		21.120			45.159			−440.423
10.5	+340.087		21.5	−108.171				
		+ 9.595			+46.589			

By interpolation from the data of this table the value of $\Delta\omega$ corresponding to the argument $31^d.81828$ is about $-2634''$. But it should be $-5898''$, consequently the mass of Titan should be changed from $\frac{1}{10000}$ to $\frac{1}{4466}$.

Having now some conception of the magnitude of the mass of Titan, it is proposed to trace the path of Hyperion from opposition to conjunction by mechanical quadratures, neglecting no powers of the disturbing forces. There are two unknown quantities to be determined: first, the velocity with which Hyperion should start from opposition; second, the mass of Titan. And there are two conditions given which suffice for their determination: first, Hyperion must arrive at conjunction with Titan after the lapse of 31.81828 days; second, it must at that time be moving at right angles to its radius vector. In order to carry out the process of mechanical quadratures we must assume the values of the two unknowns, leaving them to be corrected afterwards. I assume the velocity of Hyperion at starting from opposition to be such that it gives

$$\frac{dV}{dt} = 20^\circ.784043,$$

the unit of time being a day. This is what it would have were it moving in an elliptic orbit in which $e = 0.1$. And for the sake of a round number I shall take the mass of Titan $= \frac{1}{4500}$. The perturbations of the longitude and radius were computed by employing the indirect process. The intervals adopted at the beginning were half a day, but as the values of the functions change very rapidly near conjunction it was found expedient at the argument $27^d.75$ to reduce them to one-sixth of a day. The principal results obtained are exhibited in the following table. The perturbations, as here given, represent the deviations from the osculating ellipse at opposition. With regard to the radius, the mean distance of Titan was adopted as the unit, and, in the table, the unit of the seventh decimal of this is employed as the unit.

	$\sum \frac{d.\delta V}{dt}$	$\frac{d.\delta V}{dt}$	$\sum{}' \frac{d^2\delta r}{dt^2}$	$\sum \frac{d^2\delta r}{dt}$	$\frac{d^2\delta r}{dt^2}$
d	″	″			
0.0	0.0000			0.000	
		− 0.0541	+ 2.785		+66.210
0.5	− 0.0541			+ 66.210	
		0.4806	68.995		64.172
1.0	0.5347			130.382	
		1.2944	199.377		60.476
1.5	1.8291			190.858	
		2.4301	390.235		55.597
2.0	4.2592			246.455	
		3.8030	636.690		50.212
2.5	8.0622			296.667	
		5.3274	933.357		44.940
3.0	−13.3896			+341.607	
		− 6.9251	+1274.964		+40.289

	$\sum \frac{d.\delta V}{dt}$	$\frac{d.\delta V}{dt}$	$\sum{}^2 \frac{d^2\delta r}{dt^2}$	$\sum \frac{d^2\delta r}{dt}$	$\frac{d^2\delta r}{dt^2}$
d	″	″			
3.5	−20.3147			+381.896	
		− 8.5345	+1656.860		+36.602
4.0	28.8492			418.498	
		10.1165	2075.358		33.999
4.5	38.9657			452.497	
		11.6470	2527.855		32.520
5.0	50.6127			485.017	
		13.1162	3012.872		32.093
5.5	63.7289			517.110	
		14.5333	3529.982		32.532
6.0	78.2622			549.642	
		15.9123	4079.624		33.661
6.5	94.1745			583.303	
		17.2745	4662.927		35.282
7.0	111.4490			618.585	
		18.6460	5281.512		37.207
7.5	130.0950			655.792	
		20.0557	5937.304		39.259
8.0	150.1507			695.051	
		21.5355	6632.355		41.278
8.5	171.6862			736.329	
		23.1185	7368.684		43.117
9.0	194.8047			779.446	
		24.8431	8148.130		44.615
9.5	219.6478			824.061	
		26.7458	8972.191		45.639
10.0	246.3936			869.700	
		28.8692	9841.891		46.027
10.5	275.2628			915.727	
		31.2576	10757.618		45.606
11.0	306.5204			961.333	
		33.9591	11718.951		44.169
11.5	340.4795			1005.502	
		37.0247	12724.453		41.469
12.0	377.5042			1046.971	
		40.5078	13771.424		37.218
12.5	418.0120			1084.189	
		44.4633	14855.613		31.060
13.0	462.4753			1115.249	
		48.9447	15970.862		22.584
13.5	511.4200			1137.833	
		54.0014	17108.695		+11.317
14.0	565.4214			1149.150	
		59.6721	18257.845		− 3.250
14.5	625.0935			1145.900	
		65.9776	19403.745		21.643
15.0	691.0711			1124.257	
		72.9092	20528.002		44.335
15.5	763.9803			1079.922	
		80.4148	21607.924		71.647
16.0	844.3951			1008.275	
		88.3817	22616.199		103.657
16.5	932.7768			904.618	
		96.6138	23520.817		139.806
17.0	1029.3906			764.813	
		104.8184	24285.630		179.057
17.5	1134.2090			585.756	
		112.6022	24871.386		219.522
18.0	1246.8112			366.234	
		119.4445	25237.620		258.039
18.5	1366.2557			+108.195	
		124.7574	25345.815		290.774
19.0	1491.0131			−182.579	
		127.9111	25163.236		313.047
19.5	1618.9242			495.626	
		128.3512	24667.610		320.519
20.0	1747.2754			816.145	
		125.6494	23851.465		309.504
20.5	1872.9248			1125.649	
		119.6684	22725.816		278.431
21.0	1992.5932			1404.080	
		110.5826	21321.736		228.391
21.5	2103.1758			1632.471	
		98.9071	19689.265		163.279
22.0	2202.0829			1795.750	
		85.4130	17893.515		89.036
22.5	2287.4959			1884.786	
		71.0055	16008.729		− 12.702
23.0	2358.5014			1897.488	
		56.6601	14111.241		+ 58.537
23.5	2415.1615			1838.951	
		43.1114	12272.290		119.860
24.0	2458.2729			1719.091	
		30.9998	10553.199		167.337
24.5	−2489.2727			−1551.754	
		−20.6962	+ 9001.445		+199.326

	$\sum \frac{d.\delta V}{dt}$	$\frac{d.\delta V}{dt}$	$\sum {}^2 \frac{d^2 \delta r}{dt^2}$	$\sum \frac{d^2 \delta r}{dt}$	$\frac{d^2 \delta r}{dt^2}$
d	″	″			
25.0	−2509.9689			−1352.428	
		− 12.3526	+7649.017		+215.847
25.5	2522.3215			1136.581	
		5.9579	6512.436		217.754
26.0	2528.2794			918.827	
		− 1.3770	5593.609		206.391
26.5	2529.6564			712.436	
		+ 1.5916	4881.173		182.887
27.0	2528.0648			529.549	
		+ 3.1898	+4351.624		+148.044
27.5	−2524.8750			−381.505	
27.75	−2523.68161			−114.6686	
		+ 1.20138	+3977.6203		+11.1726
	2522.48023			103.4960	
		1.17887	3874.1243		9.1016
	2521.30136			94.3944	
		1.12578	3779.7299		6.8505
28.25	2520.17558			87.5439	
		1.04505	3692.1860		4.4043
	2519.13053			83.1396	
		0.93959	3609.0464		+ 1.7461
	2518.19094			81.3935	
		0.81271	3527.6529		− 1.1451
28.75	2517.37823			82.5386	
		0.66760	3445.1143		4.2938
	2516.71063			86.8324	
		0.50766	3358.2819		7.7282
	2516.20297			94.5606	
		0.33665	3263.7213		11.4793
29.25	2515.86632			106.0399	
		+ 0.15891	3157.6814		15.5818
	2515.70741			121.6217	
		− 0.02094	3036.0597		20.0721
	2515.72835			141.6938	
		0.19721	2894.3659		24.9862
29.75	2515.92556			166.6800	
		0.36330	2727.6859		30.3568
	2516.28886			197.0368	
		0.51126	2530.6491		36.2076
	2516.80012			233.2444	
		0.63151	2297.4047		42.5468
30.25	2517.43163			275.7912	
		0.71233	2021.6135		49.3541
	2518.14396			325.1453	
		0.73969	1696.4682		56.5685
	2518.88365			381.7138	
		0.69680	1314.7544		64.0695
30.75	2519.58045			445.7833	
		0.56393	868.9711		71.6610
	2520.14438			517.4443	
		− 0.31884	+ 351.5268		79.0582
	2520.46322			596.5025	
		+ 0.06281	− 244.9757		85.8848
31.25	2520.40041			682.3873	
		0.60583	927.3630		91.6940
	2519.79458			774.0813	
		1.33393	1701.4443		96.0099
	2518.46065			870.0912	
		2.26712	2571.5355		98.3971
31.75	2516.19353			968.4883	
		3.41866	3540.0238		98.5399
	2512.77487			1067.0282	
		+ 4.79471	4607.0520		−96.3079
	−2507.98016			−1163.3361	
			−5770.3881		

From the data of this table it is concluded by interpolation that, for the argument $31^d.81828$, the perturbations are

$$\delta V = -2513''.09, \quad \frac{d.\delta r}{dt} = -0.0006348834.$$

The unit of time for the latter is a day, and the linear unit the mean distance of Titan.

Let us suppose that the mass of Titan we have employed needs to be

multiplied by a factor μ not likely to differ much from unity, and let it be granted that within these limits the perturbations may be considered as varying proportionally to μ. Then calling ΔV the correction to the longitude of Hyperion through the change which ought to be made in the velocity attributed to it at opposition, the following equations ought to be satisfied:

$$178° 39' 9''.75 + \Delta V - 2513''.09 \mu = 178° 21' 41''.79$$
$$\frac{dr_0}{dt} - 0.0006348834 \mu = 0.$$

For convenience let it be supposed that the value of the daily mean motion, we have employed for the opposition, needs to be corrected by $60'' + \Delta n$. Then the equations may be put in the linear form.

$$26.1300 \Delta n - 2513''.09 \mu + 2614''.21 = 0$$
$$-0.004579 \Delta n - 0.6348834 \mu + 0.5682878 = 0.$$

In the coefficients of Δn is included the effect of the change in e necessary to keep $a(1-e)$ constant. It will be seen there is no leaning towards indetermination in these equations. The solution gives

$$60'' + \Delta n = + 51''.7581$$
$$\log \mu = 9.9797984.$$

The resulting mass of Titan is $m' = \frac{1}{4714}$, and the osculating elements of Hyperion at opposition are

$$\text{Daily } n = 60963''.23942$$
$$\log a = 0.0823532$$
$$e = 0.0994706.$$

The mass of Titan here arrived at is quite different from any of the values published hitherto. Prof. Newcomb's value* will, however, be in substantial agreement if it is multiplied by 3; and it appears that this ought to be done, since the number 97.4, given as the sum of 72 values, in order to obtain the mean, through some inadvertence, doubtless, has been divided by 24 instead of 72. Prof. O. Stone has deduced a larger value.† But, since its publication, he has informed me that, after the rectification of an error committed in his investigation, he arrives at a value nearly the same with mine. With regard to the value of the mass obtained by M. F. Tisserand‡ from the motion of the nodes of Iapetus, it appears difficult to explain the discrepancy, and I cannot here make the attempt.

**Astronomical Papers of the American Ephemeris*, Vol. III, p. 367.

†*Annals of Mathematics*, Vol. III, p. 161.

‡*Annales de l'Observatoire de Toulouse*, Tom. I.

From the data now in hand, without any further developments, it is possible to construct a table giving the inequality of the orbit longitude and the radius of Hyperion with the argument days after, or days yet to elapse before, opposition with Titan. Such a table follows. It corresponds to an opposition radius of 192″.582, and to the mass of Titan as here found. When the argument is days yet to elapse before opposition, the signs given in the columns headed Inequality of Orbit Longitude must be reversed.

Arg.	Ineq. of Orb.	Long.	Radius		Arg.	Ineq. of Orb.	Long.	Radius	
d	′	′	″	″	d	′	′	″	″
0.0	0.0	+115.1	192.58	+0.29	16.5	—684.7	+10.7	212.86	—3.14
0.5	+115.1	111.5	192.87	0.85	17.0	674.0	26.3	209.72	3.08
1.0	226.6	104.4	193.72	1.37	17.5	647.7	42.3	206.64	2.96
1.5	331.0	94.4	195.09	1.86	18.0	605.4	57.8	203.68	2.74
2.0	425.4	81.8	196.95	2.27	18.5	547.6	72.7	200.94	2.45
2.5	507.2	67.5	199.22	2.60	19.0	474.9	86.1	198.49	2.08
3.0	574.7	52.2	201.82	2.87	19.5	388.8	97.6	196.41	1.64
3.5	626.9	36.1	204.69	3.04	20.0	291.2	106.3	194.77	1.14
4.0	663.0	20.2	207.73	3.14	20.5	184.9	111.8	193.63	0.60
4.5	683.2	+ 4.7	210.87	3.17	21.0	— 73.1	113.8	193.03	—0.03
5.0	687.9	—10.3	214.04	3.11	21.5	+ 40.7	112.2	193.00	+0.52
5.5	677.6	24.0	217.15	3.01	22.0	152.9	107.2	193.52	1.08
6.0	653.6	36.7	220.16	2.84	22.5	260.1	98.7	194.60	1.58
6.5	616.9	48.1	223.00	2.62	23.0	358.8	87.7	196.18	2.03
7.0	568.8	58.2	225.62	2.37	23.5	446.5	74.5	198.21	2.40
7.5	510.6	66.9	227.99	2.07	24.0	521.0	59.9	200.61	2.71
8.0	443.7	74.1	230.06	1.76	24.5	580.9	44.2	203.32	2.94
8.5	369.6	80.1	231.82	1.42	25.0	625.1	28.4	206.26	3.07
9.0	289.5	84.5	233.24	1.06	25.5	653.5	+12.7	209.33	3.13
9.5	205.0	87.6	234.30	0.68	26.0	666.2	— 2.5	212.46	3.13
10.0	117.4	89.5	234.98	+0.31	26.5	663.7	16.7	215.59	3.05
10.5	+ 27.9	89.8	235.29	—0.08	27.0	647.0	30.0	218.64	2.91
11.0	— 61.9	88.9	235.21	0.46	27.5	617.0	42.1	221.55	2.73
11.5	150.8	86.4	234.75	0.83	28.0	574.9	52.9	224.28	2.49
12.0	237.2	82.8	233.92	1.20	28.5	522.0	62.2	226.77	2.21
12.5	320.0	77.8	232.72	1.56	29.0	459.8	70.1	228.98	1.91
13.0	397.8	71.2	231.16	1.88	29.5	389.7	76.9	230.89	1.58
13.5	469.0	63.4	229.28	2.19	30.0	312.8	82.1	232.47	1.21
14.0	532.4	54.2	227.09	2.47	30.5	230.7	85.8	233.68	0.84
14.5	586.6	43.6	224.62	2.70	31.0	144.9	88.1	234.52	0.45
15.0	630.2	31.6	221.92	2.90	31.5	+ 56.8	—89.2	234.97	+0.06
15.5	661.8	18.6	219.02	3.04	32.0	— 32.4		235.03	
16.0	—680.4	— 4.3	215.98	—3.12					

MEMOIR No. 46.

On Leverrier's Determination of the Second-Order Terms in the Secular Motions of the Eccentricities and Perihelia of Jupiter and Saturn.

(Astronomical Journal, Vol. IX, pp. 89–91, 1889.)

I wish to call attention to some remarkable peculiarities in the results obtained by Leverrier (*Annales de l'Observatoire de Paris, Mémoires,* Tom. X, pp. 239–260). It is well known that these terms augment the motion of the mentioned elements, which is obtained from the sole consideration of the first power of the disturbing force, by nearly a fourth part. Hence their importance from a practical point of view. The subject is treated again by Leverrier (Tom. XI, pp. 20, 23, 53, 56). Taking from the latter place the numerical data we need for our discussion, the terms involving the relative position of the planes of the orbits may be set aside as having scarcely any importance in the matter; also the few terms of the third and fourth orders with respect to the disturbing forces, which Leverrier has derived, and which scarcely augment the precision of his final results, may be neglected. For Leverrier's values of the masses let Bessel's values $m = \frac{1}{1047.879}$, $m' = \frac{1}{3501.6}$ be substituted.

With these modifications, no longer keeping separate the portions having different mass-multipliers, Leverrier's results take the reduced form of the four following differential equations which the variables e, $\tilde{\omega}$, e' and $\tilde{\omega}'$ must satisfy:—

$$
\begin{aligned}
\frac{e}{\cos\psi}\frac{d\tilde{\omega}}{dt} = & + 8''.243933\, e + 48''.7566\, e^3 + 263''.169\, ee'^2 + 2437''.73\, e^5 \\
& + 40886''.0\, e^3e'^2 + 56352''.0\, ee'^4 \\
& + \left\{ \begin{array}{l} -4''.665835\, e' - 239''.065\, e^2e' - 205''.900\, e'^3 \\ -20396''.8\, e^4e' - 101809''.8\, e^2e'^3 - 30842''.5\, e'^5 \end{array} \right\} \cos(\tilde{\omega}' - \tilde{\omega}) \\
& + \{123''.837\, ee'^2 + 27073''.9\, e^3e'^2 + 37132''.8\, ee'^4\} \cos 2(\tilde{\omega}' - \tilde{\omega}) \\
& - 11105''.2\, e^2e'^3 \cos 3(\tilde{\omega}' - \tilde{\omega}), \\
\frac{1}{\cos\psi}\frac{de}{dt} = & \left\{ \begin{array}{l} 5''.224151\, e' + 79''.688\, e^2e' + 205''.900\, e'^3 \\ + 4063''.56\, e^4e' + 33904.1\, e^2e'^3 + 30842''.5\, e'^5 \end{array} \right\} \sin(\tilde{\omega}' - \tilde{\omega}) \\
& - \{123''.837\, ee'^2 + 13516''.86\, e^3e'^2 + 37132''.8\, ee'^4\} \sin 2(\tilde{\omega}' - \tilde{\omega}) \\
& + 11105''.2\, e^2e'^3 \sin 3(\tilde{\omega}' - \tilde{\omega}), \\
\frac{e'}{\cos\psi'}\frac{d\tilde{\omega}'}{dt} = & + 18''.12312\, e' + 648''.265\, e^2e' + 828''.207\, e'^3 + 125176''.4\, e'^5
\end{aligned}
$$

$$
\begin{aligned}
&+ 277780''.7\, e^2e'^3 + 50369''.6\, e^4e' \\
&+ \left\{ \begin{matrix} - 12''.482489\, e - 196''.160\, e^3 - 1523''.643\, ee'^2 \\ - 10061''.7\, e^5 - 250947''.1\, e^3e'^2 - 380667''\, ee'^4 \end{matrix} \right\} \cos(\tilde{\omega}' - \tilde{\omega}) \\
&+ \{305''.012\, e^2e' + 183474''.2\, e^2e'^3 + 33426''.6\, e^4e'\} \cos 2(\tilde{\omega}' - \tilde{\omega}) \\
&- 27503''.5\, e^3e'^2 \cos 3(\tilde{\omega}' - \tilde{\omega}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\cos\psi'}\frac{de'}{dt} = &\left\{ \begin{matrix} - 12''.482489\, e - 196''.160\, e^3 - 507''.856\, ee'^2 \\ - 10061''.7\, e^5 - 83757''.2\, e^3e'^2 - 76591''\, ee'^4 \end{matrix} \right\} \sin(\tilde{\omega}' - \tilde{\omega}) \\
&+ \{305''.012\, e^2e' + 91976''.7\, e^2e'^3 + 33426''.6\, e^4e'\} \sin 2(\tilde{\omega}' - \tilde{\omega}) \\
&- 27503''.5\, e^3e'^2 \sin 3(\tilde{\omega}' - \tilde{\omega}).
\end{aligned}
$$

Some of the coefficients in these equations are identical, and others are seen to satisfy certain relations. To explain these, it may be remarked that when we confine our attention to the first power of the disturbing force, the second members of the equations are constant multiples of the partial derivatives of the same function R, so that representing one of the terms of R by

$$A e^i e'^{i'} \cos j(\tilde{\omega}' - \tilde{\omega}),$$

we have

$$
\begin{aligned}
\frac{e}{\cos\psi}\frac{de}{dt} &= -\frac{1}{m\sqrt{\mu a}}\frac{\partial R}{\partial \tilde{\omega}} = -\frac{1}{m\sqrt{\mu a}}\, jAe^ie'^{i'} \sin j(\tilde{\omega}' - \tilde{\omega}), \\
\frac{e}{\cos\psi}\frac{d\tilde{\omega}}{dt} &= \frac{1}{m\sqrt{\mu a}}\frac{\partial R}{\partial e} = \frac{1}{m\sqrt{\mu a}}\, iAe^{i-1}e'^{i'} \cos j(\tilde{\omega}' - \tilde{\omega}), \\
\frac{e'}{\cos\psi'}\frac{de'}{dt} &= -\frac{1}{m'\sqrt{\mu' a'}}\frac{\partial R}{\partial \tilde{\omega}'} = \frac{1}{m'\sqrt{\mu' a'}}\, jAe^ie'^{i'} \sin j(\tilde{\omega}' - \tilde{\omega}), \\
\frac{e'}{\cos\psi'}\frac{d\tilde{\omega}'}{dt} &= \frac{1}{m'\sqrt{\mu' a'}}\frac{\partial R}{\partial e'} = \frac{1}{m'\sqrt{\mu' a'}}\, i'Ae^ie'^{i'-1} \cos j(\tilde{\omega}' - \tilde{\omega}).
\end{aligned}
$$

But when we wish to add to the terms of the first order with respect to disturbing forces those of two dimensions with respect to the same quantities, the foregoing relations are no longer rigorously fulfilled, because some of the new terms result from the substitution in the portion of the perturbative function which denotes the reaction of the planet on the sun, and for which we do not pass from the value for one planet to that for the other by multiplying by a constant.

However, certain considerations connected with the possibility of having the same perturbative function for both planets, through an orthogonal transformation of variables, would seem to show that the relations given above could not be greatly disturbed.

For the purpose of exhibiting this quality from the four equations which have been given, we remark that they will furnish from one to four values for A, the coefficient of any term of R.

I have prepared the following table showing the agreement or disagree-

ment of the several values. To obtain it we make the following assumptions; let the linear unit adopted be the semi-axis major of Saturn, then the logarithm of that of Jupiter will be 9.7367410, and the mass of the Sun being denoted by unity, we shall have

$$\log\left(\frac{1}{m\sqrt{\mu a}}\right) = 3.1517336, \quad \log\left(\frac{1}{m'\sqrt{\mu' a'}}\right) = 3.5442045\,.$$

Term of R	Values of A from Equations I.	II.	III.	IV.
	$''$	$''$	$''$	$''$
Ae^2	$+\ 0.002906504$			
Ae'^2			$+\ 0.002588204$	
Ae^4	$+\ 0.00859488$			
$Ae^2e'^2$	$+\ 0.0927836$		$+\ 0.0925802$	
Ae'^4			$+\ 0.0591391$	
Ae^6	$+\ 0.28648$			
$Ae^4e'^2$	$+\ 7.2074$		$+\ 7.1934$	
$Ae^2e'^4$	$+19.8219$		$+19.8352$	
Ae'^6			$+\ 5.9589$	
$Aee' \cos(\omega' - \omega)$	$-\ 0.003289999$	$-\ 0.003683681$	$-\ 0.003565305$	$-\ 0.003565305$
$Ae^3e' \cos(\omega' - \omega)$	$-\ 0.056190$	$-\ 0.056190$	$-\ 0.056028$	$-\ 0.0056028$
$Aee'^3 \cos(\omega' - \omega)$	$-\ 0.145185$	$-\ 0.145185$	$-\ 0.145063$	$-\ 0.145063$
$Ae^5e' \cos(\omega' - \omega)$	$-\ 2.87646$	$-\ 2.86532$	$-\ 2.87387$	$-\ 2.87387$
$Ae^3e'^3 \cos(\omega' - \omega)$	-23.9296	-23.9066	-23.8922	-23.9231
$Aee'^5 \cos(\omega' - \omega)$	-21.7478	-21.7478	-21.7456	-21.8763
$Ae^2e'^2 \cos 2(\omega' - \omega)$	$+\ 0.0436603$	$+\ 0.0436603$	$+\ 0.0435595$	$+\ 0.0435595$
$Ae^4e'^2 \cos 2(\omega' - \omega)$	$+\ 4.77262$	$+\ 4.76554$	$+\ 4.77373$	$+\ 4.77373$
$Ae^2e'^4 \cos 2(\omega' - \omega)$	$+13.0916$	$+13.0916$	$+13.1012$	$+13.1354$
$Ae^3e'^3 \cos 3(\omega' - \omega)$	$-\ 2.61019$	$-\ 2.61019$	$-\ 2.61856$	$-\ 2.61856$

It will be noticed that there is approximate agreement generally between the different values. The largest discrepancy occurs in the case of the coefficient of $ee' \cos(\tilde{\omega}' - \tilde{\omega})$, where we have the anomaly of the values from the third and fourth equations agreeing, while those from the first and second are at variance. In the equations determining the elements of Saturn we have the two coefficients $-12''.482489$, $-12''.482489$, exactly identical, while, in the equations for the elements of Jupiter, the analogous coefficients $-4''.665835$, $-5''.224151$, differ. How to explain this anomaly without supposing some error in Leverrier's numbers, I cannot imagine. The details, given in Leverrier's volumes, are too slight to enable us to trace this anomaly to its origin. After transformation to our values of the masses, the several portions given for the composition of these discrepant numbers stand as follows:—

$$\begin{aligned}
&-\ \ 4''.830777 - 0''.111542 + 0''.279158 - 0''.002674 &&= -\ \ 4''.665835\,,\\
&-\ \ 4''.830777 - 0''.111542 - 0''.279158 - 0''.002674 &&= -\ \ 5''.224151\,,\\
&-11''.925816 - 0''.287959 - 0''.268714 &&= -12''.482489\,,\\
&-11''.925816 - 0''.287959 - 0''.268714 &&= -12''.482489\,.
\end{aligned}$$

Four parts are given in the case of Jupiter, while, for Saturn, there are only three. Perhaps we must suppose that the term lacking for Saturn is too insignificant to be considered. It should be noticed that, in the case of Saturn, the three portions are proportional severally to m, mm' and m^2; while, for Jupiter, the four parts are proportional severally to m', m'^2, mm' and mm'. It will be perceived that the discrepancy between the two numbers for Jupiter is owing to the quantity $0''.279158$ having opposite signs in the two equations. It does not appear easy to imagine reasons why two quantities, which are identical in the case of Saturn, should have opposite signs in the case of Jupiter. The supposition that Leverrier attributed the wrong sign to one or the other of these numbers does not seem to set matters right. The consideration of this enigma is commended to those interested in celestial mechanics.

MEMOIR No. 47.

The Secular Perturbations of Two Planets Moving in the Same Plane; With Application to Jupiter and Saturn.

(Annals of Mathematics, Vol. V, pp. 177-218, 1890.)

The solution of this problem, when we restrict ourselves to the first powers of the eccentricities, is as old as Lagrange, and is well known. Leverrier, in going over this ground, attempted to include the effect of the terms of three dimensions with respect to eccentricities and inclinations.* But when his method was applied to the four interior planets of the solar system it led to results that were nugatory. This method being that of successive approximations, the expressions for the unknowns obtained in the simplest form of the investigation were substituted in the terms of three dimensions; in consequence, he arrived at the same linear differential equations as before, but now augmented by known terms. His difficulty, in the case of the four interior planets, arose from the appearance in the results of integrating divisors which might receive very small, or even zero, values within the range of uncertainty of the values of the planetary masses.

As far as the general question is concerned, no one has attempted to push the investigation further. Under these circumstances I have thought it might be well to treat as completely as we can the very simple case where we have only two planets executing their motions in the same plane. Although we see here at a glance that the problem is reducible to quadratures, yet this taken by itself does not constitute a practical solution. Some difficulties are encountered in deriving from the quadratures series suitable for calculating the values of the unknowns. These difficulties I have succeeded in surmounting by a process which would not suggest itself, I think, at first sight.

In the application which I have made to the case of Jupiter and Saturn with neglected mutual inclination, I have carried the approximation to quantities of the fifth order, inclusive; and it is not difficult to see what must be done if it is desired to go further.

* Annales de l'Observatoire de Paris, Tom. II, pp. 105–170 and pp. [38]–[51].

I.

The first thing to be done in this investigation is to find a proper development of the potential or perturbative function. Quantities belonging to the interior planet will be denoted by symbols without an accent, and those belonging to the exterior by symbols having an accent. Let, then, m, r, a, g, u, and f denote severally the mass of the planet, the radius, the semi-axis major, the mean, eccentric, and true anomalies, while we denote the distance between the planets by Δ. The potential function Ω is then given by the double definite integral

$$\Omega = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{mm'}{\Delta}\,dg\,dg',$$

or, if the integration is accomplished with reference to the eccentric anomalies, by the double definite integral

$$\Omega = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{r}{a}\,\frac{r'}{a'}\,\frac{mm'}{\Delta}\,du\,du'.$$

These formulæ show that the potential function is proportional to the average value of the reciprocal of the distance when the mean anomalies are regarded as the independent variables, or to the average value of the product of the radii divided by the distance when the eccentric anomalies are the independent variables. As the eccentricities e and e' and the longitudes of the perihelia ϖ and ϖ' are the variable quantities whose forms as functions of the time we are seeking, it is plain they must be left indeterminate in the expression we obtain for Ω. Since Δ can be expressed in terms of u and u' as a finite form, the second formula for Ω is to be preferred.

If γ be put for $\varpi - \varpi'$, the expression for Δ, in the case we treat, is

$$\Delta = r'\left[1 - 2\frac{r}{r'}\cos(f - f' + \gamma) + \frac{r^2}{r'^2}\right]^{\frac{1}{2}}.$$

Thus, the expression for Ω becomes

$$\Omega = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{r}{aa'}\,\frac{mm'}{\left[1 - 2\frac{r}{r'}\cos(f - f' + \gamma) + \frac{r^2}{r'^2}\right]^{\frac{1}{2}}}\,du\,du'.$$

If B_j denote the same function of $\frac{r}{r'}$ that Laplace's $b_{\frac{1}{2}}^{(j)}$ is of α, the ratio of the mean distances, we may write

$$\begin{aligned}\left[1 - 2\frac{r}{r'}\cos(f - f' + \gamma) + \frac{r^2}{r'^2}\right]^{-\frac{1}{2}} &= \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} B_j \cos j(f - f' + \gamma)\\ &= \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} B_j e^{j(f-f'+\gamma)\sqrt{-1}},\end{aligned}$$

ε denoting the base of natural logarithms. If we make $\varepsilon^{u\sqrt{-1}} = s$, and put

$$\eta = \frac{1+\sqrt{1-e^2}}{2}, \qquad \omega = \frac{e}{1+\sqrt{1-e^2}},$$

from the equations

$$r = a(1 - e\cos u), \qquad r\cos f = a(\cos u - e), \qquad r\sin f = a\sqrt{1-e^2}\sin u,$$

it is easy to derive

$$r = a\eta(1-\omega s)\left(1 - \frac{\omega}{s}\right),$$

$$\varepsilon^{f\sqrt{-1}} = \frac{s-\omega}{1-\omega s}.$$

Thus

$$\frac{r'}{\Delta} = \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} B_j\left(\frac{s-\omega}{1-\omega s}\right)^j\left(\frac{s-\omega'}{1-\omega' s'}\right)^{-j}\varepsilon^{j\gamma\sqrt{-1}}.$$

Seeking now an expression for B_j in terms of s and s', we have

$$(1 - 2\alpha\cos\varphi + \alpha^2)^{-\frac{1}{2}} = \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} b^{(j)}\varepsilon^{j\varphi\sqrt{-1}},$$

(we omit Laplace's subscript $\frac{1}{2}$, as it is unnecessary for the purposes of distinction). We can regard $b^{(j)}$ as an approximate value of B_j, and the true value can be developed in a convergent series by Maclaurin's Theorem, if the perihelion radius of the exterior planet always exceeds the aphelion radius of the interior; that is, if

$$\frac{a'e' + ae}{a' - a} < 1.$$

The augmentation which α receives is

$$\frac{r}{r'} - \alpha = \alpha\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} - \alpha.$$

Thus

$$B_j = \sum_{i=0}^{i=+\infty}\frac{1}{i!}\alpha^i\frac{d^i b^{(j)}}{d\alpha^i}\left[\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} - 1\right]^i.$$

Expanding the latter factor by the binomial theorem,

$$B_j = \sum_{i=0}^{i=+\infty}\sum_{k=0}^{k=i}\frac{(-1)^{i-k}}{k!(i-k)!}\alpha^i\frac{d^i b^{(j)}}{d\alpha^i}\left[\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)}\right]^k.$$

Substituting this value of B_j in the expression given above for $\frac{r'}{\Delta}$, and multiplying the result by

$$\frac{mm'r}{aa'} = \frac{mm'}{a'}\eta(1-\omega s)\left(1-\frac{\omega}{s}\right),$$

and employing the symbol ∇ to denote the operation of taking the coefficient of $s^0 s'^0$ in the development of a function of s and s' in a series of integral powers and products of s and s', we shall have

$$\Omega = \frac{mm'}{2a'}\sum_{j=-\infty}^{j=+\infty}\sum_{i=0}^{i=+\infty}\sum_{k=0}^{k=i}\frac{(-1)^{i-k}}{k!\,(i-k)!}a^i\frac{d^ib^{(j)}}{da^i}\eta^{k+1}\eta'^{-k}\varepsilon^{j\gamma\sqrt{-1}}$$
$$\times\nabla\left[s^j s'^{-j}(1-\omega s)^{k-j+1}\left(1-\frac{\omega}{s}\right)^{k+j+1}(1-\omega' s')^{j-k}\left(1-\frac{\omega'}{s'}\right)^{-k-j}\right].$$

Let us put

$$E_i^{(j)} = \eta^i\nabla\left[s^j(1-\omega s)^{i-j}\left(1-\frac{\omega}{s}\right)^{i+j}\right].$$

This quantity is then a function of e. Let $E'^{(j)}_i$ be the same function of e' that $E^{(j)}_i$ is of e. Then we can write

$$\Omega = \frac{mm'}{2a'}\sum_{j=-\infty}^{j=+\infty}\sum_{i=0}^{i=+\infty}\sum_{k=0}^{k=i}\frac{(-1)^{i-k}}{k!(i-k)!}a^i\frac{d^ib^{(j)}}{da^i}E^{(j)}_{k+1}E'^{(j)}_{-k}\varepsilon^{j\gamma\sqrt{-1}}.$$

This constitutes the infinite series to be employed in this investigation, and it remains only to study the properties of the functions of e denoted by $E^{(j)}_i$. By expanding the binomial factors involved in $E^{(j)}_i$ and performing the operation denoted by ∇, we shall get

$$E_i^{(j)} = (-1)^j\frac{(i+1)(i+2)\ldots(i+j)}{1.2\ldots j}\eta^i\omega^j$$
$$\times\left[1+\frac{i-j}{1}\frac{i}{j+1}\omega^2+\frac{(i-j)(i-j-1)}{1.2}\frac{i(i-1)}{(j+1)(j+2)}\omega^4+\ldots\right].$$

The series within the brackets is a case of the hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots,$$

treated by Gauss in a memoir entitled "Disquisitiones generales circa seriem infinitam, etc."* This series gives the value of $E^{(j)}_i$ in terms of η and ω, but it may readily be transformed into another expressed in terms of e. Adopting Gauss's notation for this species of series

$$E_i^{(j)} = (-1)^j\frac{(i+1)(i+2)\ldots(i+j)}{1.2\ldots j}\eta^i\omega^j F(j-i,-i,j+1,\omega^2).$$

*See Gauss, Werke, Band III, p. 123.

But from Gauss's equation [100], p. 225 of the volume quoted,

$$F(j-i, -i, j+1, \omega^2) = (1+\omega^2)^{i-j} F\left(\frac{j-i}{2}, \frac{j-i+1}{2}, j+1, \frac{4\omega^2}{(1+\omega^2)^2}\right),$$

and

$$e^2 = \frac{4\omega^2}{(1+\omega^2)^2}.$$

In consequence

$$\begin{aligned} E_i^{(j)} &= \frac{(i+j)!}{i!\,j!}\left(-\frac{e}{2}\right)^j F\left(\frac{j-i}{2}, \frac{j-i+1}{2}, j+1, e^2\right) \\ &= \frac{(i+1)\ldots(i+j)}{1\ldots j}\left(-\frac{e}{2}\right)^j \Big[1 + \frac{(i-j)(i-j-1)}{1.(j+1)}\left(\frac{e}{2}\right)^2 \\ &\qquad + \frac{(i-j)(i-j-1)(i-j-2)(i-j-3)}{1.2.(j+1)(j+2)}\left(\frac{e}{2}\right)^4 + \ldots\Big]. \end{aligned}$$

It is remarkable that when i and j are integers the value of $E_i^{(j)}$ is equivalent to a rational function of the two quantities e and $\sqrt{1-e^2}$. For, when i is a positive integer, the series first given terminates after a finite number of terms. The same thing occurs in the second series when $i-j$ is not negative. By Gauss's equation [82], p. 209 of the volume quoted,

$$F\left(\frac{j+i}{2}, \frac{j+i+1}{2}, j+1, e^2\right) = (1-e^2)^{-\frac{2i-1}{2}} F\left(\frac{j-i+2}{2}, \frac{j-i+1}{2}, j+1, e^2\right).$$

From this it follows that

$$\begin{aligned} E_{-i}^{(j)} &= \frac{(i-1)(i-2)\ldots(i-j)}{1.2\ldots j}\left(\frac{e}{2}\right)^j (1-e^2)^{-\frac{2i-1}{2}} F\left(\frac{j-i+2}{2}, \frac{j-i+1}{2}, j+1, e^2\right) \\ &= \frac{(i-1)\ldots(i-j)}{1.2..j}\left(\frac{e}{2}\right)^j (1-e^2)^{-\frac{2i-1}{2}}\Big[1 + \frac{(i-j-1)(i-j-2)}{1.(j+1)}\left(\frac{e}{2}\right)^2 \\ &\qquad + \frac{(i-j-1)\ldots(i-j-4)}{1.2.(j+1)(j+2)}\left(\frac{e}{2}\right)^4 + \ldots\Big], \end{aligned}$$

which affords a finite expression for $E_i^{(j)}$ when i is negative. It will be noticed that $E_i^{(j)} = 0$, when i, not zero, is not greater than j.

In order that the symmetry of the expression for Ω may be seen, we will write the development of this quantity at length without the employment of the summatory signs:

$$\begin{aligned} \Omega = \frac{mm'}{2a'}\Big\{ &\quad b^{(0)} E_1^{(0)} E'^{(0)}_0 \\ &- a\frac{db^{(0)}}{da}[E_1^{(0)} E'^{(0)}_0 - E_2^{(0)} E'^{(0)}_{-1}] \\ &+ \tfrac{1}{2} a^2 \frac{d^2b^{(0)}}{da^2}[E_1^{(0)} E'^{(0)}_0 - 2E_2^{(0)} E'^{(0)}_{-1} + E_3^{(0)} E'^{(0)}_{-2}] \\ &- \frac{1}{2.3} a^3 \frac{d^3b^{(0)}}{da^3}[E_1^{(0)} E'^{(0)}_0 - 3E_2^{(0)} E'^{(0)}_{-1} + 3E_3^{(0)} E'^{(0)}_{-2} - E_4^{(0)} E'^{(0)}_{-3}] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \Big\} \end{aligned}$$

$$+\frac{mm'}{a'}\left\{\begin{array}{l}\text{Same expression as above, except that } b,\ E, \text{ and}\\ E' \text{ now take 1 as the upper index instead of 0.}\end{array}\right\}\cos\gamma$$

$$+\frac{mm'}{a'}\left\{\begin{array}{l}\text{Same expression, except that } b,\ E, \text{ and } E'\\ \text{now take 2 as the upper index instead of 0.}\end{array}\right\}\cos 2\gamma$$

$$+\frac{mm'}{a'}\left\{\begin{array}{l}\text{Same expression, except that } b,\ E, \text{ and } E\\ \text{now take 3 as the upper index instead of 0.}\end{array}\right\}\cos 3\gamma$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It may be noticed that the terms in $E_2^{(1)}E'^{(1)}_{-1}$, $E_2^{(2)}E'^{(2)}_{-1}$, $E_3^{(2)}E'^{(2)}_{-2}$, $E_2^{(3)}E'^{(3)}_{-1}$, $E_3^{(3)}E'^{(3)}_{-2}$, $E_4^{(3)}E'^{(3)}_{-3}$, etc., can be omitted in writing the expression, as the latter factors of these products vanish. However, the symmetry is more apparent when they are retained.

The following table exhibits the values of all the E's required in developing Ω to the terms of the sixth order, inclusive. They are expressed as functions of e, and the finite form is given as perhaps more interesting than the development in ascending powers of e.

$$
\begin{array}{ll}
E_1^{(0)} = 1 & E_0^{(0)} = 1\\
E_2^{(0)} = 1 + \frac{1}{2}e^2 & E_{-1}^{(0)} = (1-e^2)^{-\frac{1}{2}}\\
E_3^{(0)} = 1 + \frac{3}{2}e^2 & E_{-2}^{(0)} = (1-e^2)^{-\frac{3}{2}}\\
E_4^{(0)} = 1 + 3e^2 + \frac{3}{8}e^4 & E_{-3}^{(0)} = [1 + \frac{1}{2}e^2](1-e^2)^{-\frac{5}{2}}\\
E_5^{(0)} = 1 + 5e^2 + \frac{15}{8}e^4 & E_{-4}^{(0)} = [1 + \frac{3}{2}e^2](1-e^2)^{-\frac{7}{2}}\\
E_6^{(0)} = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6 & E_{-5}^{(0)} = [1 + 3e^2 + \frac{3}{8}e^4](1-e^2)^{-\frac{9}{2}}\\
E_7^{(0)} = 1 + \frac{21}{2}e^2 + \frac{105}{8}e^4 + \frac{35}{16}e^6 & E_{-6}^{(0)} = [1 + 5e^2 + \frac{15}{8}e^4](1-e^2)^{-\frac{11}{2}}
\end{array}
$$

$$
\begin{array}{ll}
E_1^{(1)} = -e & E_0^{(1)} = -\dfrac{1-\sqrt{1-e^2}}{e}\\
E_2^{(1)} = -\frac{3}{2}e & E_{-1}^{(1)} = 0\\
E_3^{(1)} = -2e - \frac{1}{2}e^3 & E_{-2}^{(1)} = \frac{1}{2}e(1-e^2)^{-\frac{3}{2}}\\
E_4^{(1)} = -\frac{5}{2}e - \frac{15}{8}e^3 & E_{-3}^{(1)} = e(1-e^2)^{-\frac{5}{2}}\\
E_5^{(1)} = -3e - \frac{9}{2}e^3 - \frac{3}{8}e^5 & E_{-4}^{(1)} = [\frac{3}{2}e + \frac{3}{8}e^3](1-e^2)^{-\frac{7}{2}}\\
E_6^{(1)} = -\frac{7}{2}e - \frac{35}{4}e^3 - \frac{35}{16}e^5 & E_{-5}^{(1)} = [2e + \frac{3}{2}e^3](1-e^2)^{-\frac{9}{2}}\\
E_7^{(1)} = -4e - 15e^3 - \frac{15}{2}e^5 - \frac{5}{16}e^7 & E_{-6}^{(1)} = [\frac{5}{2}e + \frac{15}{4}e^3 + \frac{5}{16}e^5](1-e^2)^{-\frac{11}{2}}
\end{array}
$$

$$
\begin{array}{ll}
E_1^{(2)} = \frac{3}{2}(1-\sqrt{1-e^2}) - \frac{1}{2}\dfrac{(1-\sqrt{1-e^2})^3}{e} & E_0^{(2)} = \dfrac{(1-\sqrt{1-e^2})^2}{e^2}\\
E_2^{(2)} = \frac{3}{2}e^2 & E_{-1}^{(2)} = 0\\
E_3^{(2)} = \frac{5}{2}e^2 & E_{-2}^{(2)} = 0\\
E_4^{(2)} = \frac{15}{4}e^2 + \frac{5}{8}e^4 & E_{-3}^{(2)} = \frac{1}{4}e^2(1-e^2)^{-\frac{5}{2}}\\
E_5^{(2)} = \frac{21}{4}e^2 + \frac{21}{8}e^4 & E_{-4}^{(2)} = \frac{3}{4}e^2(1-e^2)^{-\frac{7}{2}}\\
E_6^{(2)} = 7e^2 + 7e^4 + \frac{7}{16}e^6 & E_{-5}^{(2)} = [\frac{3}{2}e^2 + \frac{1}{4}e^4](1-e^2)^{-\frac{9}{2}}\\
E_7^{(2)} = 9e^2 + 15e^4 + \frac{45}{16}e^6 & E_{-6}^{(2)} = [\frac{5}{2}e^2 + \frac{5}{4}e^4](1-e^2)^{-\frac{11}{2}}
\end{array}
$$

$$
\begin{array}{ll}
E_1^{(3)} = -2\dfrac{(1-\sqrt{1-e^2})^2}{e} + \dfrac{(1-\sqrt{1-e^2})^4}{e^3} & E_0^{(3)} = -\dfrac{(1-\sqrt{1-e^2})^3}{e^3}\\
E_2^{(3)} = -\frac{5}{4}e(1-\sqrt{1-e^2}) + \frac{5}{4}\dfrac{(1-\sqrt{1-e^2})^3}{e} & E_{-1}^{(3)} = 0\\
\qquad - \frac{1}{4}\dfrac{(1-\sqrt{1-e^2})^5}{e^3} &
\end{array}
$$

$$
\begin{aligned}
E_3^{(3)} &= -5e^3 & E_{-2}^{(3)} &= 0\\
E_4^{(3)} &= -\tfrac{35}{4}e^3 & E_{-3}^{(3)} &= 0\\
E_5^{(3)} &= -14e^3 - \tfrac{7}{4}e^5 & E_{-4}^{(3)} &= \tfrac{1}{8}e^3(1-e^2)^{-\frac{5}{2}}\\
E_6^{(3)} &= -21e^3 - \tfrac{63}{8}e^5 & E_{-5}^{(3)} &= \tfrac{1}{2}e^3(1-e^2)^{-\frac{7}{2}}\\
E_7^{(3)} &= -30e^3 - \tfrac{45}{2}e^5 - \tfrac{9}{8}e^7 & E_{-6}^{(3)} &= [\tfrac{5}{4}e^3 + \tfrac{5}{32}e^5](1-e^2)^{-\frac{9}{2}}.
\end{aligned}
$$

In the present investigation it will be more convenient to make use of a development of $E_i^{(j)}$ in powers of $\sqrt{\left(\frac{1-\sqrt{1-e^2}}{2}\right)} = \theta$. By substituting in the formula for $E_i^{(j)}$ in terms of e the values

$$
\left(\frac{e}{2}\right)^2 = \theta^2 - \theta^4,
$$

$$
\left(\frac{e}{2}\right)^j = \theta^j(1-\theta^2)^{\frac{j}{2}},
$$

making, for the sake of brevity, $i - j = k$, and carrying the development to terms of the sixth order, inclusive, we obtain

$$
\begin{aligned}
E_i^{(j)} = (-1)^j \frac{(i+1)\dots(i+j)}{1\dots j}\theta^j \Big\{ 1 &+ \left[\frac{k(k-1)}{1.(j+1)} - \frac{j}{2}\right]\theta^2\\
&+ \left[\frac{k(k-1)(k-2)(k-3)}{1.2.(j+1)(j+2)} - \frac{j+2}{2}\,\frac{k(k-1)}{1.(j+1)} + \frac{j(j-2)}{2.4}\right]\theta^4\\
&+ \left[\frac{k(k-1)(k-2)(k-3)(k-4)(k-5)}{1.2.3.(j+1)(j+2)(j+3)} - j\frac{j+4}{2}\,\frac{k(k-1)(k-2)(k-3)}{1.2.(j+1)(j+2)}\right.\\
&\qquad \left. + j\frac{(j+2)}{2.4}\,\frac{k(k-1)}{1.(j+1)} - \frac{j(j-2)(j-4)}{2.4.6}\right]\theta^6 \Big\}.
\end{aligned}
$$

Or, particularizing with respect to j,

$$
\begin{aligned}
E_i^{(0)} &= 1 + i(i-1)\theta^2 + i(i-1)\left[\frac{(i-2)(i-3)}{2.2} - 1\right]\theta^4\\
&\qquad + \frac{i(i-1)(i-2)(i-3)}{1.2.1.2}\left[\frac{(i-4)(i-5)}{3.3} - 2\right]\theta^6,\\
E_i^{(1)} &= -(i+1)\theta\Big\{1 + \left[\frac{(i-1)(i-2)}{1.2} - \frac{1}{2}\right]\theta^2\\
&\qquad + \left[\frac{(i-1)(i-2)(i-3)(i-4)}{1.2.2.3} - \frac{3}{2}\,\frac{(i-1)(i-2)}{1.2} - \frac{1.1}{2.4}\right]\theta^4\Big\}.\\
E_i^{(2)} &= \frac{(i+1)(i+2)}{1.2}\theta^2\left\{1 + \left[\frac{(i-2)(i-3)}{1.3} - 1\right]\theta^2\right\},\\
E_i^{(3)} &= \frac{(i+1)(i+2)(i+3)}{1.2.3}\theta^3.
\end{aligned}
$$

And, specializing still further,

$$E_1^{(0)} = 1 \qquad E_0^{(0)} = 1$$
$$E_2^{(0)} = 1 + 2\theta^2 - 2\theta^4 \qquad E_{-1}^{(0)} = 1 + 2\theta^2 + 4\theta^4 + 8\theta^6$$
$$E_3^{(0)} = 1 + 6\theta^2 - 6\theta^4 \qquad E_{-2}^{(0)} = 1 + 6\theta^2 + 24\theta^4 + 80\theta^6$$
$$E_4^{(0)} = 1 + 12\theta^2 - 6\theta^4 - 12\theta^6 \qquad E_{-3}^{(0)} = 1 + 12\theta^2 + 78\theta^4 + 380\theta^6$$
$$E_5^{(0)} = 1 + 20\theta^2 + 10\theta^4 - 60\theta^6 \qquad E_{-4}^{(0)} = 1 + 20\theta^2 + 190\theta^4 + 1260\theta^6$$
$$E_6^{(0)} = 1 + 30\theta^2 + 60\theta^4 - 160\theta^6 \qquad E_{-5}^{(0)} = 1 + 30\theta^2 + 390\theta^4 + 3360\theta^6$$
$$E_7^{(0)} = 1 + 42\theta^2 + 168\theta^4 - 280\theta^6 \qquad E_{-6}^{(0)} = 1 + 42\theta^2 + 714\theta^4 + 7728\theta^6$$
$$E_1^{(1)} = -\theta[2 - \theta^2 - \tfrac{1}{4}\theta^4] \qquad E_0^{(1)} = -\theta[1 + \tfrac{1}{2}\theta^2 + \tfrac{3}{8}\theta^4]$$
$$E_3^{(1)} = -\theta[4 + 2\theta^2 - \tfrac{13}{2}\theta^4] \qquad E_{-2}^{(1)} = \theta[1 + \tfrac{11}{2}\theta^2 + \tfrac{167}{8}\theta^4]$$
$$E_4^{(1)} = -\theta[5 + \tfrac{25}{2}\theta^2 - \tfrac{185}{8}\theta^4] \qquad E_{-3}^{(1)} = \theta[2 + 19\theta^2 + \tfrac{489}{4}\theta^4]$$
$$E_5^{(1)} = -\theta[6 + 33\theta^2 - \tfrac{171}{4}\theta^4] \qquad E_{-4}^{(1)} = \theta[3 + \tfrac{87}{2}\theta^2 + \tfrac{2817}{8}\theta^4]$$
$$E_6^{(1)} = -\theta[7 + \tfrac{133}{2}\theta^2 - \tfrac{287}{8}\theta^4] \qquad E_{-5}^{(1)} = \theta[4 + 82\theta^2 + \tfrac{1763}{2}\theta^4]$$
$$E_7^{(1)} = -\theta[8 + 116\theta^2 + 59\theta^4] \qquad E_{-6}^{(1)} = \theta[5 + \tfrac{275}{2}\theta^2 = \tfrac{15115}{8}\theta^4]$$
$$E_1^{(2)} = \theta^2[3 - \theta^2] \qquad E_0^{(2)} = \theta^2[1 + \theta^2]$$
$$E_4^{(2)} = \theta^2[15 - 5\theta^2] \qquad E_{-3}^{(2)} = \theta^2[1 + 9\theta^2]$$
$$E_5^{(2)} = \theta^2[21 + 21\theta^2] \qquad E_{-4}^{(2)} = \theta^2[3 + 39\theta^2]$$
$$E_6^{(2)} = \theta^2[28 + 84\theta^2] \qquad E_{-5}^{(2)} = \theta^2[6 + 106\theta^2]$$
$$E_7^{(2)} = \theta^2[36 + 204\theta^2] \qquad E_{-6}^{(2)} = \theta^2[10 + 230\theta^2]$$
$$E_1^{(3)} = -4\theta^3 \qquad E_0^{(3)} = -\theta^3$$
$$E_5^{(3)} = -56\theta^3 \qquad E_{-4}^{(3)} = \theta^3$$
$$E_6^{(3)} = -84\theta^3 \qquad E_{-5}^{(3)} = 4\theta^3$$
$$E_7^{(3)} = -120\theta^3 \qquad E_{-6}^{(3)} = 10\theta^3$$

Through multiplication we obtain

$$E_1^{(0)}E'^{(0)}_0 = 1$$
$$E_2^{(0)}E'^{(0)}_{-1} = 1 + 2\theta^2 + 2\theta'^2 - 2\theta^4 + 4\theta^2\theta'^2 + 4\theta'^4 + 0\theta^6 - 4\theta^4\theta'^2 + 8\theta^2\theta'^4 + 8\theta'^6$$
$$E_3^{(0)}E'^{(0)}_{-2} = 1 + 6\theta^2 + 6\theta'^2 - 6\theta^4 + 36\theta^2\theta'^2 + 24\theta'^4 + 0\theta^6 - 36\theta^4\theta'^2 + 144\theta^2\theta'^4 + 80\theta'6 -$$
$$E_4^{(0)}E'^{(0)}_{-3} = 1 + 12\theta^2 + 12\theta'^2 - 6\theta^4 + 144\theta^2\theta'^2 + 78\theta'^4 - 12\theta^6 - 72\theta^4\theta'^2 + 936\theta^2\theta'^4 + 380\theta'^6$$
$$E_5^{(0)}E'^{(0)}_{-4} = 1 + 20\theta^2 + 20\theta'^2 + 10\theta^4 + 400\theta^2\theta'^2 + 190\theta'^4$$
$$- 60\theta^6 + 200\theta^4\theta'^2 + 3800\theta^2\theta'^4 + 1260\theta'^6$$
$$E_6^{(0)}E'^{(0)}_{-5} = 1 + 30\theta^2 + 30\theta'^2 + 60\theta^4 + 900\theta^2\theta'^2 + 390\theta'^4$$
$$- 160\theta^6 + 1800\theta^4\theta'^2 + 11700\theta^2\theta'^4 + 3360\theta'^6$$
$$E_7^{(0)}E'^{(0)}_{-6} = 1 + 42\theta^2 + 42\theta'^2 + 168\theta^4 + 1764\theta^2\theta'^2$$
$$+ 714\theta'^4 - 280\theta^6 + 7056\theta^4\theta'^2 + 29988\theta^2\theta'^4 + 7728\theta'^6$$
$$E_1^{(1)}E'^{(1)}_0 = \theta\theta'[2 - \theta^2 + \theta'^2 - \tfrac{1}{4}\theta^4 - \tfrac{1}{2}\theta^2\theta'^2 + \tfrac{3}{4}\theta'^4]$$
$$E_3^{(1)}E'^{(1)}_{-2} = \theta\theta'[-4 - 2\theta^2 - 22\theta'^2 + \tfrac{13}{2}\theta^4 - 11\theta^2\theta'^2 - \tfrac{167}{2}\theta'^4]$$
$$E_4^{(1)}E'^{(1)}_{-3} = \theta\theta'[-10 - 25\theta^2 - 95\theta'^2 + \tfrac{185}{4}\theta^4 - \tfrac{475}{2}\theta^2\theta'^2 - \tfrac{2195}{4}\theta'^4]$$
$$E_5^{(1)}E'^{(1)}_{-4} = \theta\theta'[-18 - 99\theta^2 - 261\theta'^2 + \tfrac{513}{4}\theta^4 - \tfrac{2871}{2}\theta^2\theta'^2 - \tfrac{8451}{4}\theta'^4]$$
$$E_6^{(1)}E'^{(1)}_{-5} = \theta\theta'[-28 - 266\theta^2 - 574\theta'^2 + \tfrac{287}{2}\theta^4 - 5453\theta^2\theta'^2 - \tfrac{12841}{2}\theta'^4]$$
$$E_7^{(1)}E'^{(1)}_{-6} = \theta\theta'[-40 - 580\theta^2 - 1100\theta'^2 - 295\theta^4 - 15950\theta^2\theta'^2 - 15115\theta'^4]$$
$$E_1^{(2)}E'^{(2)}_0 = \theta^2\theta'^2[3 - \theta^2 + 3\theta'^2]$$
$$E_4^{(2)}E'^{(2)}_{-3} = \theta^2\theta'^2[15 - 5\theta^2 + 135\theta'^2]$$
$$E_5^{(2)}E'^{(2)}_{-4} = \theta^2\theta'^2[63 + 63\theta^2 + 819\theta'^2]$$
$$E_6^{(2)}E'^{(2)}_{-5} = \theta^2\theta'^2[168 + 504\theta^2 + 2968\theta'^2]$$
$$E_7^{(2)}E'^{(2)}_{-6} = \theta^2\theta'^2[360 + 2040\theta^2 + 8280\theta'^2]$$

$$E_1^{(3)} E'^{(3)}_{0} = 4\theta^3\theta'^3$$
$$E_5^{(3)} E'^{(3)}_{-4} = -56\theta^3\theta'^3$$
$$E_6^{(3)} E'^{(3)}_{-5} = -336\theta^3\theta'^3$$
$$E_7^{(3)} E'^{(3)}_{-6} = -1200\theta^3\theta'^3.$$

If, in the expression for Ω, we call the function of the eccentricities which multiplies $\frac{(-1)^i}{i!}\alpha^i \frac{d^ib^{(j)}}{da^i}$ in the coefficient of $\cos j\gamma$, $M_i^{(j)}$, and Δ denoting the characteristic of finite differences with respect to the variable i, it will be seen that we have

$$\Delta^n M_0^{(j)} = (-1)^n E_{n+1}^{(j)} E'^{(j)}_{-n}.$$

Then the expressions for $M_i^{(j)}$ can be derived by considering the preceding expressions, taken alternately with the positive and negative sign, as the successive differences of these functions with respect to the index i; and it will be advantageous to apply the process separately to each power and product of θ and θ'. The exhibition of this follows:—

Coefficients of $\cos 0\gamma$:

Coefficients of θ^0.

1						
	−1					
0		+1				
	0		−1			
0		0		+1		
	0		0		−1	
0		0		0		+1
	0		0		0	
0		0		0		
	0		0			
0		0				
	0					
0						

Coefficients of θ^2 and θ'^2.

0						
	−2					
−2		+6				
	+4		−12			
+2		−6		+20		
	−2		+8		−30	
0		+2		−10		+42
	0		−2		+12	
0		0		+2		
	0		0			
0		0				
	0					
0						

Coefficients of θ^4.

0						
	+2					
+2		−6				
	−4		+6			
−2		0		+10		
	−4		+16		−60	
−6		+16		−50		+168
	+12		−34		+108	
+6		−18		+58		
	−6		+24			
0		+6				
	0					
0						

Coefficients of $\theta^2\theta'^2$.

0						
	− 4					
− 4		+ 36				
	+ 32		− 144			
+ 28		− 108		+ 400		
	− 76		+ 256		− 900	
− 48		+ 148		− 500		+ 1764
	+ 72		− 244		+ 864	
+ 24		− 96		+ 364		
	− 24		+ 120			
0		+ 24				
	0					
0						

Coefficients of θ'^4.

0						
	− 4					
− 4		+ 24				
	+ 20		− 78			
+ 16		− 54		+ 190		
	− 34		+ 112		− 390	
− 18		+ 58		− 200		+ 714
	+ 24		− 88		+ 324	
6		− 30		+ 124		
	− 6		+ 36			
0		+ 6				
	0					
0						

Coefficients of θ^6.

0						
	0					
0		0				
	0		+ 12			
0		+ 12		− 60		
	+ 12		− 48		+ 160	
+ 12		− 36		+ 100		− 280
	− 24		+ 52		− 120	
− 12		+ 16		− 20		
	− 8		+ 32			
− 20		+ 48				
	+ 40					
+ 20						

Coefficients of $\theta^4\theta'^2$.

0						
	+ 4					
+ 4		− 36				
	− 32		+ 72			
− 28		+ 36		+ 200		
	+ 4		+ 272		− 1800	
− 24		+ 308		− 1600		+ 7056
	+ 312		− 1328		+ 5256	
+ 288		− 1020		+ 3656		
	− 708		+ 2328			
− 420		+ 1308				
	+ 600					
+ 180						

Coefficients of $\theta^2\theta'^4$.

0						
	− 8					
− 8		+ 144				
	+ 136		− 936			
+ 128		− 792		+ 3800		
	− 656		+ 2864		− 11700	
− 528		+ 2072		− 7900		+ 29988
	+ 1416		− 5036		+ 18288	
+ 888		− 2964		+ 10388		
	− 1548		+ 5352			
− 660		+ 2388				
	+ 840					
+ 180						

Coefficients of θ'^6.

0						
	− 8					
− 8		+ 80				
	+ 72		− 380			
+ 64		− 300		+ 1260		
	− 228		+ 880		− 3360	
− 164		+ 580		− 2100		+ 7728
	+ 352		− 1220		+ 4368	
+ 188		− 640		+ 2268		
	− 288		+ 1048			
− 100		+ 408				
	+ 120					
+ 20						

Coefficients multiplying $\theta\theta' \cos\gamma$:

Coefficients of θ^0.

+ 2						
	0					
+ 2		− 4				
	− 4		+ 10			
− 2		+ 6		− 18		
	+ 2		− 8		+ 28	
0		− 2		+ 10		− 40
	0		+ 2		− 12	
0		0		− 2		
	0		0			
0		0				
	0					
0						

Coefficients of θ^2.

− 1						
	0					
− 1		− 2				
	− 2		+ 25			
− 3		+ 23		− 99		
	+ 21		− 74		+ 266	
+ 18		− 51		+ 167		− 580
	− 30		+ 93		− 314	
− 12		+ 42		− 147		
	+ 12		− 54			
0		− 12				
	0					
0						

Coefficients of θ'^2.

+ 1						
	0					
+ 1		− 22				
	− 22		+ 95			
− 21		+ 73		− 261		
	+ 51		− 166		+ 574	
+ 30		− 93		+ 313		− 1100
	− 42		+ 147		− 526	
− 12		+ 54		− 213		
	+ 12		− 66			
0		− 12				
	0					
0						

Coefficients of θ^4 multiplied by 4.

− 1						
	0					
− 1		+ 26				
	+ 26		− 185			
+ 25		− 159		+ 513		
	− 133		+ 328		− 574	
− 108		+ 169		− 61		− 1180
	+ 36		+ 267		− 1754	
− 72		+ 436		− 1815		
	+ 472		− 1548			
+ 400		− 1112				
	− 640					
− 240						

Coefficients of $\theta^2\theta'^2$ multiplied by 2.

− 1						
	0					
− 1		− 22				
	− 22		+ 475			
− 23		+ 453		− 2871		
	+ 431		− 2396		+ 10906	
+ 408		− 1943		+ 8035		− 31900
	− 1512		+ 5639		− 20994	
− 1104		+ 3696		− 12959		
	+ 2184		− 7320			
+ 1080		− 3624				
	− 1440					
− 360						

Coefficients of θ'^4 multiplied by 4.

+ 3						
	0					
+ 3		− 334				
	− 334		+ 2195			
− 331		+ 1861		− 8451		
	+ 1527		− 6256		+ 24682	
+ 1196		− 4395		+ 16231		− 60460
	− 2868		+ 9975		− 35778	
− 1672		+ 5580		− 19547		
	+ 2712		− 9572			
+ 1040		− 3992				
	− 1280					
− 240						

Coefficients multiplying $\theta^2\theta'^2 \cos 2\gamma$:

Coefficients of θ^0.

+ 3						
	0					
+ 3		0				
	0		− 15			
+ 3		− 15		+ 63		
	− 15		+ 48		− 168	
− 12		+ 33		− 105		+ 360
	+ 18		− 57		+ 192	
+ 6		− 24		+ 87		
	− 6		+ 30			
0		+ 6				
	0					
0						

Coefficients of θ^2.

− 1						
	0					
− 1		0				
	0		+ 5			
− 1		+ 5		+ 63		
	+ 5		+ 68		− 504	
+ 4		+ 73		− 441		+ 2040
	+ 78		− 373		+ 1536	
+ 82		− 300		+ 1095		
	− 222		+ 722			
− 140		+ 422				
	+ 200					
+ 60						

Coefficients of θ'^2.

+ 3						
	0					
+ 3		0				
	0		− 135			
+ 3		− 135		+ 819		
	− 135		+ 684		− 2968	
− 132		+ 549		− 2149		+ 8280
	+ 414		− 1465		+ 5312	
+ 282		− 916		+ 3163		
	− 502		+ 1698			
− 220		+ 782				
	+ 280					
+ 60						

Coefficients of $\theta^3\theta'^3 \cos 3\gamma$.

+ 4						
	0					
+ 4		0				
	0		0			
+ 4		0		− 56		
	0		− 56		+ 336	
+ 4		− 56		+ 280		− 1200
	− 56		+ 224		− 864	
− 52		+ 168		− 584		
	+ 112		− 360			
+ 60		− 192				
	− 80					
− 20						

We can now write the explicit development of Ω as follows:

$$
\begin{aligned}
\frac{a'}{mm'}\Omega = \tfrac{1}{2}\Big\{ \quad & b^{(0)} \\
& + a\frac{db^{(0)}}{da}[2\theta^2 + 2\theta'^2 - 2\theta^4 + 4\theta^2\theta'^2 + 4\theta'^4 + 0\theta^6 - 4\theta^4\theta'^2 + 8\theta^2\theta'^4 + 8\theta'^6] \\
& + \tfrac{1}{2}a^2\frac{d^2b^{(0)}}{da^2}[2\theta^2 + 2\theta'^2 - 2\theta^4 + 28\theta^2\theta'^2 + 16\theta'^4 \\
& \qquad\qquad + 0\theta^6 - 28\theta^4\theta'^2 + 128\theta^2\theta'^4 + 64\theta'^6] \\
& + \frac{1}{2.3}a^3\frac{d^3b^{(0)}}{da^3}[6\theta^4 + 48\theta^2\theta'^2 + 18\theta'^4 - 12\theta^6 \\
& \qquad\qquad + 24\theta^4\theta'^2 + 528\theta^2\theta'^4 + 164\theta'^6] \\
& + \frac{1}{2.3.4}a^4\frac{d^4b^{(0)}}{da^4}[6\theta^4 + 24\theta^2\theta'^2 + 6\theta'^4 - 12\theta^6 \\
& \qquad\qquad + 288\theta^4\theta'^2 + 888\theta^2\theta'^4 + 188\theta'^6] \\
& + \frac{1}{2.3.4.5}a^5\frac{d^5b^{(0)}}{da^5}[20\theta^6 + 420\theta^4\theta'^2 + 660\theta^2\theta'^4 + 100\theta'^6] \\
& + \frac{1}{2.3.4.5.6}a^6\frac{d^6b^{(0)}}{da^6}[20\theta^6 + 180\theta^4\theta'^2 + 180\theta^2\theta'^4 + 20\theta'^6]\Big\} \\
& + \Big\{\Big(b^{(1)} - a\frac{db^{(1)}}{da}\Big)\Big[2 - \theta^2 + \theta'^2 - \tfrac{1}{4}\theta^4 - \tfrac{1}{2}\theta^2\theta'^2 + \tfrac{3}{4}\theta'^4\Big] \\
& \quad - \tfrac{1}{2}a^2\frac{d^2b^{(1)}}{da^2}\Big[2 + 3\theta^2 + 21\theta'^2 - \tfrac{25}{4}\theta^4 + \tfrac{23}{2}\theta^2\theta'^2 + \tfrac{221}{4}\theta'^4\Big] \\
& \quad - \frac{1}{2.3}a^3\frac{d^3b^{(1)}}{da^3}[18\theta^2 + 30\theta'^2 - 27\theta^4 + 204\theta^2\theta'^2 + 299\theta'^4] \\
& \quad - \frac{1}{2.3.4}a^4\frac{d^4b^{(1)}}{da^4}[12\theta^2 + 12\theta'^2 + 18\theta^4 + 552\theta^2\theta'^2 + 418\theta'^4] \\
& \quad - \frac{1}{2.3.4.5}a^5\frac{d^5b^{(1)}}{da^5}[100\theta^4 + 540\theta^2\theta'^2 + 260\theta'^4] \\
& \quad - \frac{1}{2.3.4.5.6}a^6\frac{d^6b^{(1)}}{da^6}[60\theta^4 + 180\theta^2\theta'^2 + 60\theta'^4]\Big\}\,\theta\theta'\cos\gamma \\
& + \Big\{\Big(b^{(2)} - a\frac{db^{(2)}}{da} + \tfrac{1}{2}a^2\frac{d^2b^{(2)}}{da^2}\Big)[3 - \theta^2 + 3\theta'^2] \\
& \qquad\qquad + \frac{1}{2.3}a^3\frac{d^3b^{(2)}}{da^3}[12 - 4\theta^2 + 132\theta'^2] \\
& \qquad\qquad + \frac{1}{2.3.4}a^4\frac{d^4b^{(2)}}{da^4}[6 - 82\theta^2 + 282\theta'^2] \\
& \qquad\qquad + \frac{1}{2.3.4.5}a^5\frac{d^5b^{(2)}}{da^5}[140\theta^2 + 220\theta'^2] \\
& \qquad\qquad + \frac{1}{2.3.4.5.6}a^6\frac{d^6b^{(2)}}{da^6}[60\theta^2 + 60\theta'^2]\Big\}\,\theta^2\theta'^2\cos 2\gamma \\
& + \Big\{4b^{(3)} - 4a\frac{db^{(3)}}{da} + 2a^2\frac{d^2b^{(3)}}{da^2} - \tfrac{2}{3}a^3\frac{d^3b^{(3)}}{da^3} - \tfrac{1}{6}a^4\frac{d^4b^{(3)}}{da^4} \\
& \qquad\qquad - \tfrac{1}{2}a^5\frac{d^5b^{(3)}}{da^5} - \tfrac{1}{36}a^6\frac{d^6b^{(3)}}{da^6}\Big\}\,\theta^3\theta'^3\cos 3\gamma.
\end{aligned}
$$

In order to have as few functions of α to deal with as possible, we gather together all the terms having the same powers of θ and θ' as factors. Also it will serve our purposes better to have the development of Ω in powers of $\cos\gamma$ than in cosines of multiples of γ. For convenience in writing we denote $\alpha^i \dfrac{d^i b^{(j)}}{d\alpha^i}$ by (j, i). We then put

$$
\begin{aligned}
A_1^{(0)} &= (0,1) + \tfrac{1}{2}(0,2),\\
A_2^{(0)} &= -(0,1) - \tfrac{1}{2}(0,2) + \tfrac{1}{2}(0,3) + \tfrac{1}{8}(0,4),\\
A_3^{(0)} &= 2(0,1) + 7(0,2) + 4(0,3) + \tfrac{1}{2}(0,4) - 3(2,0) + 3(2,1)\\
&\quad - \tfrac{3}{2}(2,2) - 2(2,3) - \tfrac{1}{4}(2,4),\\
A_4^{(0)} &= 2(0,1) + 4(0,2) + \tfrac{3}{2}(0,3) + \tfrac{1}{8}(0,4),\\
A_5^{(0)} &= -(0,3) - \tfrac{1}{4}(0,4) + \tfrac{1}{12}(0,5) + \tfrac{1}{72}(0,6),\\
A_6^{(0)} &= -2(0,1) - 7(0,2) + 2(0,3) + 6(0,4) + \tfrac{7}{4}(0,5) + \tfrac{1}{8}(0,6)\\
&\quad + (2,0) - (2,1) + \tfrac{1}{2}(2,2) + \tfrac{2}{3}(2,3) - \tfrac{41}{12}(2,4) - \tfrac{7}{6}(2,5) - \tfrac{1}{12}(2,6),\\
A_7^{(0)} &= 4(0,1) + 32(0,2) + 44(0,3) + \tfrac{27}{2}(0,4) + \tfrac{11}{4}(0,5) + \tfrac{1}{8}(0,6)\\
&\quad - 3(2,0) + 3(2,1) - \tfrac{3}{2}(2,2) - 22(2,3) - \tfrac{47}{4}(2,4) - \tfrac{11}{6}(2,5) - \tfrac{1}{12}(2,6),\\
A_8^{(0)} &= 4(0,1) + 16(0,2) + \tfrac{41}{8}(0,3) + \tfrac{47}{12}(0,4) + \tfrac{5}{12}(0,5) + \tfrac{1}{72}(0,6),\\
A_0^{(1)} &= 2(1,0) - 2(1,1) - (1,2),\\
A_1^{(1)} &= -(1,0) + (1,1) - \tfrac{3}{2}(1,2) - 3(1,3) - \tfrac{1}{2}(1,4),\\
A_2^{(1)} &= (1,0) - (1,1) - \tfrac{21}{2}(1,2) - 5(1,3) - \tfrac{1}{2}(1,4),\\
A_3^{(1)} &= -\tfrac{1}{4}(1,0) + \tfrac{1}{4}(1,1) + \tfrac{25}{8}(1,2) + \tfrac{3}{2}(1,3) - \tfrac{3}{4}(1,4) - \tfrac{5}{6}(1,5) - \tfrac{1}{12}(1,6),\\
A_4^{(1)} &= -\tfrac{1}{2}(1,0) + \tfrac{1}{2}(1,1) - \tfrac{23}{4}(1,2) - 34(1,3) - 23(1,4) - \tfrac{3}{2}(1,5) - \tfrac{1}{4}(1,6)\\
&\quad - 12(3,0) + 12(3,1) - 6(3,2) + 2(3,3) + \tfrac{13}{2}(3,4) + \tfrac{3}{2}(3,5) + \tfrac{1}{12}(3,6),\\
A_5^{(1)} &= \tfrac{3}{4}(1,0) - \tfrac{3}{4}(1,1) - \tfrac{281}{8}(1,2) - \tfrac{292}{6}(1,3) - \tfrac{209}{12}(1,4) - \tfrac{13}{6}(1,5) - \tfrac{1}{12}(1,6),\\
\tfrac{1}{2}A_0^{(2)} &= 3(2,0) - 3(2,1) + \tfrac{3}{2}(2,2) + 2(2,3) + \tfrac{1}{4}(2,4),\\
\tfrac{1}{2}A_1^{(2)} &= -(2,0) + (2,1) - \tfrac{1}{2}(2,2) - \tfrac{2}{3}(2,3) + \tfrac{41}{12}(2,4) + \tfrac{7}{6}(2,5) + \tfrac{1}{12}(2,6),\\
\tfrac{1}{2}A_2^{(2)} &= 3(2,0) - 3(2,1) + \tfrac{3}{2}(2,2) + 22(2,3) + \tfrac{47}{4}(2,4) + \tfrac{11}{6}(2,5) + \tfrac{1}{12}(2,6),\\
\tfrac{1}{4}A_0^{(3)} &= 4(3,0) - 4(3,1) + 2(3,2) - \tfrac{2}{3}(3,3) - \tfrac{13}{6}(3,4) - \tfrac{1}{2}(3,5) - \tfrac{1}{36}(3,6).
\end{aligned}
$$

Then, neglecting the term which is independent of θ, θ', and γ for the reason that it is useless for our purposes, we shall have

$$
\begin{aligned}
\frac{a'}{mm'}\Omega = A_1^{(0)}(\theta^2 + \theta'^2) + A_2^{(0)}\theta^4 + A_3^{(0)}\theta^2\theta'^2 + A_4^{(0)}\theta'^4 + A_5^{(0)}\theta^6&\\
+ A_6^{(0)}\theta^4\theta'^2 + A_7^{(0)}\theta^2\theta'^4 + A_8^{(0)}\theta'^6&\\
+ [A_0^{(1)} + A_1^{(1)}\theta^2 + A_2^{(1)}\theta'^2 + A_3^{(1)}\theta^4 + A_4^{(1)}\theta^2\theta'^2 + A_5^{(1)}\theta'^4]\,\theta\theta'\cos\gamma&\\
+ [A_0^{(2)} + A_1^{(2)}\theta^2 + A_2^{(2)}\theta'^2]\,\theta^2\theta'^2\cos^2\gamma&\\
+ A_0^{(3)}\theta^3\theta'^3\cos^3\gamma.&
\end{aligned}
$$

In order to make an application of the method to the case of Jupiter and Saturn, we take from Runkle's Tables of the Coefficients of the Pertur-

bative Function the values of log (j, i) corresponding to the argument $\log \alpha = 9.7367414$.

i.	$j=0$.	$j=1$.	$j=2$.	$j=3$.
0	0.3385227	9.7929622	9.4112303	9.0721143
1	9.6447549	9.9080135	9.7803244	9.5982418
2	9.9323686	9.8807530	0.0203420	0.0219693
3	0.2943862	0.3204279	0.3188228	0.3995660
4	0.8737099	0.8712079	0.8884960	0.9011936
5	1.5571487	1.5610571	1.5658243	1.5798073
6	2.3402885	2.3412199	2.3462289	2.3533961

Making use of these values, we obtain for this special case

$$\begin{aligned}\frac{a'}{mm'}\,\Omega = {} & 0.8692176\,(\theta^2 + \theta'^2) + 1.05019\theta^4 + 11.85269\theta^2\theta'^2 + 8.19486\theta'^4 \\ & + 2.207\theta^6 + 46.126\theta^4\theta'^2 + 157.464\theta^2\theta'^4 + 89.730\theta'^6 \\ & -[1.1365062 + 10.94248\theta^2 + 22.34085\theta'^2 + 42.355\theta^4 \\ & \qquad + 335.361\theta^2\theta'^2 + 362.413\theta'^4]\,\theta\theta' \cos\gamma \\ & + 2[6.63740 + 86.288\theta^2 + 223.228\theta'^2]\,\theta^2\theta'^2 \cos^2\gamma \\ & - 172.837\theta^3\theta'^3 \cos^3\gamma.^*\end{aligned}$$

II.

The portion of the subject which treats of the integration of certain differential equations is now to be attended to. Denoting the mass of the sun by M, and putting

$$\mu = M + m, \quad \mu' = M + m', \quad G = m\sqrt{\mu a}\sqrt{1-e^2}, \quad G' = m'\sqrt{\mu' a'}\sqrt{1-e'^2},$$

the differential equations which determine the eccentricities and positions of the perihelia of the two planets are

$$\frac{dG}{dt} = \frac{d\Omega}{d\tilde{\omega}}, \qquad \frac{d\tilde{\omega}}{dt} = -\frac{d\Omega}{dG},$$
$$\frac{dG'}{dt} = \frac{d\Omega}{d\tilde{\omega}'}, \qquad \frac{d\tilde{\omega}'}{dt} = -\frac{d\Omega}{dG'}.$$

But since Ω involves $\tilde{\omega}$ and $\tilde{\omega}'$ only through $\gamma = \tilde{\omega} - \tilde{\omega}'$, we have

$$\frac{d\Omega}{d\tilde{\omega}} + \frac{d\Omega}{d\tilde{\omega}'} = 0.$$

Hence

$$G + G' = \text{a constant}$$

is an integral of the problem. This integral equation may be more suitably expressed in terms of the variables θ and θ' which we have before employed.

* An error which affects the last two lines of this formula in the original memoir is corrected here. Many of the following numbers are, to some extent vitiated by this, but I have not thought it worth while to recompute them.

Then K denoting an arbitrary constant, and denoting the constant quantities $m\sqrt{\mu a}$, $m'\sqrt{\mu' a'}$ by $\frac{1}{\lambda^2}$, $\frac{1}{\lambda'^2}$,

$$\frac{\theta^2}{\lambda^2} + \frac{\theta'^2}{\lambda'^2} = K.$$

The value of K is ascertained by substituting in the left member of this equation for θ and θ' the values they have at a definite epoch. We can now reduce the number of variables in the problem from four to three by adopting a variable ν to replace θ and θ', such that

$$\theta = \lambda\sqrt{K}\sin\tfrac{1}{2}\nu, \qquad \theta' = \lambda'\sqrt{K}\cos\tfrac{1}{2}\nu.$$

$\frac{1}{2}\nu$ remains always in the first quadrant. Denoting the angles of the eccentricities by ϕ and ϕ', the eccentricities are determined by the formulæ

$$e = \sin\varphi, \qquad e' = \sin\varphi,$$
$$\sin\tfrac{1}{2}\varphi = \lambda\sqrt{K}\sin\tfrac{1}{2}\nu, \qquad \sin\tfrac{1}{2}\varphi' = \lambda'\sqrt{K}\cos\tfrac{1}{2}\nu.$$

Making the substitutions in Ω necessary to make it involve ν instead of θ and θ', we put

$$\theta^2 = \tfrac{1}{2}\lambda^2 K(1-\cos\nu), \qquad \theta'^2 = \tfrac{1}{2}\lambda' K(1+\cos\nu), \qquad \theta\theta' = \tfrac{1}{2}\lambda\lambda' K\sin\nu.$$

The function Ω becomes, then, divisible by K, and, in order to simplify, we shall put $\Omega = KR$. Therefore, if we write x for $\cos\nu$ and put

$$B_0^{(0)} = \frac{mm'}{a'}\Big\{\tfrac{1}{2}(\lambda^2 + \lambda'^2)A_1^{(0)} + \tfrac{1}{4}(\lambda^4 A_2^{(0)} + \lambda^2\lambda'^2 A_3^{(0)} + \lambda'^4 A_4^{(0)})K$$
$$+ \tfrac{1}{8}(\lambda^6 A_5^{(0)} + \lambda^4\lambda'^2 A_6^{(0)} + \lambda^2\lambda'^4 A_7^{(0)} + \lambda'^6 A_8^{(0)})K^2\Big\},$$

$$B_1^{(0)} = \frac{mm'}{a'}\Big\{\tfrac{1}{2}(-\lambda^2 + \lambda'^2)A_1^{(0)} - \tfrac{1}{2}(\lambda^4 A_2^{(0)} - \lambda'^4 A_4^{(0)})K$$
$$+ \tfrac{1}{8}(-3\lambda^6 A_5^{(0)} - \lambda^4\lambda'^2 A_6^{(0)} + \lambda^2\lambda'^4 A_7^{(0)} + 3\lambda'^6 A_8^{(0)})K^2\Big\},$$

$$B_2^{(0)} = \frac{mm'}{a'}\Big\{\tfrac{1}{4}(\lambda^4 A_2^{(0)} - \lambda^2\lambda'^2 A_3^{(0)} + \lambda'^4 A_4^{(0)})K$$
$$+ \tfrac{1}{8}(3\lambda^6 A_5^{(0)} - \lambda^4\lambda'^2 A_6^{(0)} - \lambda^2\lambda'^4 A_7^{(0)} + 3\lambda'^6 A_8^{(0)})K^2\Big\},$$

$$B_3^{(0)} = \frac{mm'}{a'}\Big\{\tfrac{1}{8}(-\lambda^6 A_5^{(0)} + \lambda^4\lambda'^2 A_6^{(0)} - \lambda^2\lambda'^4 A_7^{(0)} + \lambda'^6 A_8^{(0)})K^2\Big\},$$

$$B_0^{(1)} = \frac{mm'}{a'}\frac{\lambda\lambda'}{2}\Big\{A_0^{(1)} + \tfrac{1}{2}(\lambda^2 A_1^{(1)} + \lambda'^2 A_2^{(1)})K + \tfrac{1}{4}(\lambda^4 A_3^{(1)} + \lambda^2\lambda'^2 A_4^{(1)} + \lambda'^4 A_5^{(1)})K^2\Big\},$$

$$B_1^{(1)} = \frac{mm'}{a'}\frac{\lambda\lambda'}{2}\Big\{\tfrac{1}{2}(-\lambda^2 A_1^{(1)} + \lambda'^2 A_2^{(1)})K + \tfrac{1}{2}(-\lambda^4 A_3^{(1)} + \lambda'^4 A_5^{(1)})K^2\Big\},$$

$$B_2^{(1)} = \frac{mm'}{a'}\frac{\lambda\lambda'}{2}\Big\{\tfrac{1}{4}(\lambda^4 A_3^{(1)} - \lambda^2\lambda'^2 A_4^{(1)} + \lambda'^4 A_5^{(1)})K^2\Big\},$$

$$B_0^{(2)} = \frac{mm'}{a'}\frac{\lambda^2\lambda'^2}{4}\Big\{A_0^{(2)}K + \tfrac{1}{2}(\lambda^2 A_1^{(2)} + \lambda'^2 A_2^{(2)})K^2\Big\},$$

$$B_1^{(2)} = \frac{mm'}{a'}\frac{\lambda^2\lambda'^2}{8}(-\lambda^2 A_1^{(2)} + \lambda'^2 A_2^{(2)})K^2,$$

$$B_0^{(3)} = \frac{mm'}{a'}\frac{\lambda^3\lambda'^3}{8}A_0^{(3)}K^2;$$

we shall then have

$$\begin{aligned} R = B_0^{(0)} &+ B_1^{(0)}x + B_2^{(0)}x^2 + B_3^{(0)}x^3 + \dots \\ &+ [B_0^{(1)} + B_1^{(1)}x + B_2^{(1)}x^2 + \dots] \sin\nu \cos\gamma \\ &+ [B_0^{(2)} + B_1^{(2)}x + \dots] \sin^2\nu \cos^2\gamma \\ &+ [B_0^{(3)} + \dots] \sin^3\nu \cos^3\gamma \\ &+ \dots\dots\dots\dots \end{aligned}$$

With this expression for R it is readily seen from the preceding differential equations that the differential equation determining ν is

$$\frac{d\nu}{dt} = -\frac{1}{\sin\nu}\frac{dR}{d\gamma},$$

or

$$\frac{dx}{dt} = \frac{dR}{d\gamma}.$$

Since $R =$ a constant is evidently an integral of the problem, we shall have

$$\frac{dR}{d\nu}\frac{d\nu}{dt} + \frac{dR}{d\gamma}\frac{d\gamma}{dt} = 0.$$

Whence is derived

$$\frac{d\gamma}{dt} = \frac{1}{\sin\nu}\frac{dR}{d\nu}.$$

We still need an additional equation giving the value of some other function of $\tilde{\omega}$ and $\tilde{\omega}'$ than $\tilde{\omega} - \tilde{\omega}'$. If we select $\tilde{\omega} + \tilde{\omega}'$ we have

$$\frac{d(\tilde{\omega} + \tilde{\omega}')}{dt} = -\frac{dQ}{dG} - \frac{dQ}{dG'}.$$

If K is kept evident in the expressions for the various B's, so that the partial derivatives of them with respect to this quantity may be taken, we shall have

$$\frac{dQ}{dG} = \frac{d(KR)}{dK}\frac{dK}{dG} + K\frac{dR}{d\nu}\frac{d\nu}{dG} = -\tfrac{1}{2}\frac{d(KR)}{dK} - \frac{1}{2\tan\frac{\nu}{2}}\frac{dR}{d\nu},$$

$$\frac{dQ}{dG'} = \frac{d(KR)}{dK}\frac{dK}{dG'} + K\frac{dR}{d\nu}\frac{d\nu}{dG'} = -\tfrac{1}{2}\frac{d(KR)}{dK} + \tfrac{1}{2}\tan\frac{\nu}{2}\frac{dR}{d\nu}.$$

Whence

$$\begin{aligned} \frac{d(\tilde{\omega} + \tilde{\omega}')}{dt} &= \frac{d(KR)}{dK} + \frac{\cos\nu}{\sin\nu}\frac{dR}{d\nu} \\ &= \frac{d(KR)}{dK} + \cos\nu\frac{d\gamma}{dt}. \end{aligned}$$

In making our numerical application we take the mean distance a' as the unit, when a becomes the same as α previously given, and assume for the masses the values

$$m = \frac{1}{1047.879}, \qquad m' = \frac{1}{3482.2}.$$

These give

$$\log \lambda = 1.5758667, \qquad \log \lambda' = 1.7708956.$$

The values adopted for the eccentricities at the beginning of 1850 are

$$e = 0.04825801, \qquad e' = 0.05606467.$$

These furnish the equations

$$\theta = [8.4778154] \sin \frac{\nu}{2}, \qquad \theta' = [8.6728444] \cos \frac{\nu}{2},$$

and the function R becomes

$$\begin{aligned} R = {} & 0.0005906465 + 0.0002543964x + 0.00000196780x^2 \\ & + 0.000000019394x^3 \\ & - [0.0003548741 + 0.00000629406x + 0.00000008731x^2] \sin \nu \cos \gamma \\ & + [0.00000148778 + 0.00000004479x] \sin^2\nu \cos^2\gamma \\ & - 0.000000006560 \sin^3\nu \cos^3\gamma. \end{aligned}$$

The value of the constant in the integral equation

$$R = C$$

is ascertained by substituting in the expression for R the values which ν and γ have at a definite epoch, as 1850. C being determined, the equation $R = C$ can be solved, regarding $\sin \nu \cos \gamma$ as the quantity whose value is to be obtained. This value can be supposed developed in powers of $\cos \nu = x$, and we write

$$\sin \nu \cos \gamma = H = D_0 + D_1 x + D_2 x^2 + D_3 x^3 + \ldots .$$

The readiest method of obtaining the D's is by substituting the last expression in R and then equating the resulting coefficients of each power of x to zero. We thus have a system of equations determining the D's. These can be solved by successive approximation. If C is allowed to appear as an indeterminate in the expressions for the D's, H can be partially differentiated with reference to this quantity.

We can now make H play the rôle of R; for we have

$$\frac{dx}{dt} = \frac{dR}{d\gamma}, \qquad \frac{d\gamma}{dt} = -\frac{dR}{dx}, \qquad \text{and} \qquad \gamma = \text{arc cos} \frac{H}{\sqrt{1 - x^2}}.$$

Thus

$$\frac{dt}{dx} = \frac{d\gamma}{dC} = -\frac{\frac{dH}{dC}}{\sqrt{1-x^2-H^2}},$$

where the radical in the denominator must receive the sign of $\sin\gamma$; for we have

$$\begin{aligned} \cos\nu &= x, \\ \sin\nu\cos\gamma &= H = D_0 + D_1 x + D_2 x^2 + D_3 x^3 + \ldots, \\ \sin\nu\sin\gamma &= \sqrt{1-x^2-H^2}. \end{aligned}$$

If we suppose the orbits are always ellipses x cannot pass the limits ± 1. Thus x must oscillate between a maximum and a minimum value, while dH/dC remains constantly of the same sign. The maximum and minimum values of x are evidently the two consecutive real roots of the equation in x

$$1 - x^2 - H^2 = 0,$$

which contain between them at any time the actual value of x. Calling these roots a and b, we may write

$$1 - x^2 - H^2 = (\mathrm{a} - x)(x - \mathrm{b})\,Q,$$

where Q is positive for all values of x lying between a and b; and when the eccentricities are always small, the variation of Q is slight in comparison with its magnitude. In the place of x we can adopt a new variable, ψ, such that

$$x = \frac{\mathrm{a}+\mathrm{b}}{2} - \frac{\mathrm{a}-\mathrm{b}}{2}\cos\psi.$$

Then

$$\frac{dx}{\sqrt{(\mathrm{a}-x)(x-\mathrm{b})}} = d\psi,$$

and the differential equation giving ψ in terms of t is

$$\frac{dt}{d\psi} = -\frac{\frac{dH}{dC}}{\sqrt{Q}}.$$

To see how all this applies in the case of Jupiter and Saturn we assume the following values of the longitudes of the perihelia at the epoch 1850.0:

$$\tilde{\omega} = 11^\circ\ 54'\ 31''.18, \qquad \tilde{\omega}' = 90^\circ\ 6'\ 57''.55.$$

The value of the constant C being now determined, and the equation $R = C$ modified in such a way that it becomes more suitable for solution, we have

$$\begin{aligned} 0.4021256 = &- 0.7168638x - 0.0055451x^2 - 0.0000546x^3 \\ &+ [1 + 0.0177360x + 0.0002460x^2]\sin\nu\cos\gamma \\ &- [0.0041924 + 0.0001262x]\sin^2\nu\cos^2\gamma \\ &+ 0.0000185\sin^3\nu\cos^3\gamma. \end{aligned}$$

When this equation is solved with reference to $\sin\nu\cos\gamma$ as the unknown, we obtain

$$H = 0.4028046 + 0.7121389x - 0.0050141x^2 - 0.0000050x^3.$$

And when we ascertain what increment H receives from an infinitesimal increment in the quantity C, it results that

$$-\frac{dH}{dC} = 2827.425 - 33.179x + 0.005x^2.$$

The equation $1 - x^2 - H^2 = 0$, in this case, is

$$0.8377485 - 0.5737057x - 1.5031028x^2 + 0.0071447x^3 - 0.0000180x^4 = 0.$$

The consecutive real roots of this which contain between them the value of x at 1850.0 are

$$\mathrm{a} = 0.5803236, \qquad \mathrm{b} = -0.9586738.$$

We derive from these the limiting values of ν, which are

$$54^\circ\ 31'\ 36''.14 \qquad \text{and} \qquad 163^\circ\ 28'\ 14''.01.$$

Thus, when $\gamma = 0^\circ$, the minimum e of Jupiter has place, which is 0.02752623; as also the maximum e' of Saturn, which is 0.08362800. And, when $\gamma = 180^\circ$, the maximum e of Jupiter has place, and is 0.05944555; and the minimum e' of Saturn, which is 0.01353514.

The remaining factor of the equation, two of whose roots we have just obtained, is

$$Q = 1.5058180 - 0.0071522x + 0.0000180x^2.$$

Whence

$$\frac{1}{\sqrt{Q}} = 0.8149177 + 0.0019353x + 0.0000020x^2.$$

Substituting, then, for x the expression

$$x = -0.1891751 - 0.7694987\cos\psi,$$

we get

$$\begin{aligned}\frac{dt}{d\psi} &= 2304.1185 - 21.5662x - 0.0543x^2\\ &= 2308.1802 + 16.5794\cos\psi - 0.0161\cos 2\psi.\end{aligned}$$

Integrating this, c being the arbitrary constant,

$$t + c = 2308.1802\psi + 16.5794\sin\psi - 0.0080\sin 2\psi.$$

Inverting this series and changing the numerical coefficients into seconds of arc we get

$$\begin{aligned}\psi = 19''.05825(t + c) - 1481''.57\sin[19''.05825(t + c)]\\ + 6''.04\sin 2[19''.05825(t + c)].\end{aligned}$$

From the value which ψ must have at the epoch 1850.0, t being counted thence,

$$19''.05825c = 277^\circ\ 9'\ 9''.15.$$

Also, we have

$$\begin{aligned}
\cos\nu &= -\ 0.1891751 - 0.7694987\cos\psi,\\
\sin\nu\cos\gamma &= +\ 0.2679063 - 0.5494490\cos\psi - 0.0029673\cos^2\psi\\
&\quad + 0.0000023\cos^3\psi,\\
\sin\nu\sin\gamma &= [0.9446898 + 0.0017265\cos\psi + 0.0000018\cos^2\psi]\sin\psi.
\end{aligned}$$

These equations enable us to determine the eccentricities and difference of the longitudes of the perihelia at any given time.

It remains to find the longitudes of the perihelia themselves. We have

$$\frac{d\tilde{\omega}}{dt} = \tfrac{1}{2}C + \tfrac{1}{2}K\frac{dR}{dK} + \frac{1+x}{2}\frac{d\gamma}{dt},$$

$$\frac{d\tilde{\omega}'}{dt} = \tfrac{1}{2}C + \tfrac{1}{2}K\frac{dR}{dK} - \frac{1-x}{2}\frac{d\gamma}{dt}.$$

Or

$$\frac{d(\tilde{\omega} - \frac{1}{2}Ct)}{dx} = -\tfrac{1}{2}K\frac{d\gamma}{dK} + \frac{1+x}{2}\frac{d\gamma}{dx},$$

$$\frac{d(\tilde{\omega}' - \frac{1}{2}Ct)}{dx} = -\tfrac{1}{2}K\frac{d\gamma}{dK} - \frac{1-x}{2}\frac{d\gamma}{dx}.$$

Or

$$\frac{d(\tilde{\omega} - \frac{1}{2}Ct)}{dx} = \tfrac{1}{2}\,\frac{K\dfrac{dH}{dK} - (1+x)\dfrac{dH}{dx} - \dfrac{Hx}{1-x}}{\sqrt{(1-x^2-H^2)}},$$

$$\frac{d(\tilde{\omega}' - \frac{1}{2}Ct)}{dx} = \tfrac{1}{2}\,\frac{K\dfrac{dH}{dK} + (1-x)\dfrac{dH}{dx} + \dfrac{Hx}{1+x}}{\sqrt{(1-x^2-H^2)}}.$$

Here K must be left indeterminate in the coefficients D_0, D_1, etc., of H, in order that we may get $\dfrac{dH}{dK}$. In the next place, we derive

$$\frac{d(\tilde{\omega} - \frac{1}{2}Ct)}{d\psi} = \tfrac{1}{2}\,\frac{K\dfrac{dH}{dK} - (1+x)\dfrac{dH}{dx} - \dfrac{Hx}{1-x}}{\sqrt{Q}},$$

$$\frac{d(\tilde{\omega}' - \frac{1}{2}Ct)}{d\psi} = \tfrac{1}{2}\,\frac{K\dfrac{dH}{dK} + (1-x)\dfrac{dH}{dx} + \dfrac{Hx}{1+x}}{\sqrt{Q}}.$$

When H, which is an infinite series in integral powers of x, is divided by $1 - x$ or $1 + x$, remainders independent of x are left over which are equivalent to

what H becomes when in it we make $x = 1$ and $x = -1$. These remainders we denote as $H(1)$ and $H(-1)$. Then we may write

$$\frac{d(\tilde{\omega} - \frac{1}{2}Ct)}{d\psi} = \frac{1}{2}\,\frac{-\dfrac{H(1)}{1-x} + \sum\limits_{i=0}^{i=\infty}\left[K\dfrac{dD_i}{dK} - (i-1)D_i - iD_{i+1} + D_{i+2} + D_{i+3} + \dots\right]x^i}{\sqrt{Q}},$$

$$\frac{d(\tilde{\omega}' - \frac{1}{2}Ct)}{d\psi} = \frac{1}{2}\,\frac{-\dfrac{H(-1)}{1+x} + \sum\limits_{i=0}^{i=\infty}\left[K\dfrac{dD_i}{dK} - (i-1)D_i + iD_{i+1} + D_{i+2} - D_{i+3} + \dots\right]x^i}{\sqrt{Q}}.$$

The difference of these equations gives

$$\frac{d\gamma}{d\psi} = \frac{-\frac{1}{2}\dfrac{H(1)}{1-x} + \frac{1}{2}\dfrac{H(-1)}{1+x} + \sum\limits_{i=0}^{i=\infty}[-iD_{i+1} + D_{i+3} + D_{i+5}]x^i}{\sqrt{Q}}.$$

Since γ returns to the same value after ψ has augmented by a circumference it follows that when the right member is expanded in an infinite series containing, besides two terms in the form of fractions having $1 - x$ and $1 + x$ as denominators, a set of terms proceeding according to cosines of multiples of ψ, the coefficient of the zero multiple of ψ must vanish. This is not immediately evident from the form of the expression. Hence I proceed to prove it to the degree of approximation we adopt. Let

$$\frac{1}{\sqrt{Q}} = E_0 + E_1 x + E_2 x^2 + \dots;$$

then, omitting the two terms in the form of fractions and having $1 - x$ and $1 + x$ for denominators, it will be perceived that we have

$$\frac{d\gamma}{d\psi} = D_3E_0 + (D_0 + D_2)E_1 + D_1E_2 - [D_2E_0 - D_0E_2]x - [2D_3E_0 - D_2E_1]x^2.$$

Substituting for x its value in terms of ψ, if our proposition is true we ought to have

$$D_3E_0 + (D_0 + D_2)E_1 + D_1E_2 - [D_2E_0 - D_0E_2]\frac{\mathrm{a} + \mathrm{b}}{2}$$
$$- [2D_3E_0 - D_2E_1]\left[\tfrac{3}{2}\left(\frac{\mathrm{a} + \mathrm{b}}{2}\right)^2 - \tfrac{1}{2}\mathrm{ab}\right] = 0.$$

But if

$$Q = M_0 + M_1 x + M_2 x^2 + \dots,$$
$$E_0 = M_0^{-\frac{1}{2}}, \qquad E_1 = -\tfrac{1}{2}M_0^{-\frac{3}{2}}M_1, \qquad E_2 = -\tfrac{1}{2}M_0^{-\frac{3}{2}}M_2 + \tfrac{3}{8}M_0^{-\frac{5}{2}}M_1^2,$$

and M_0, M_1, M_2, a, and b are determined by the equations

$$\begin{aligned}
\mathrm{ab}M_0 &= D_0^2 - 1, \\
(\mathrm{a} + \mathrm{b})M_0 - \mathrm{ab}M_1 &= -2D_0D_1, \\
M_0 - (\mathrm{a} + \mathrm{b})M_1 + \mathrm{ab}M_2 &= 1 + D_1^2 + 2D_0D_2, \\
M_1 - (\mathrm{a} + \mathrm{b})M_2 &= 2(D_1D_2 + D_0D_3), \\
M_2 &= D_2^2 + 2D_1D_3.
\end{aligned}$$

By substituting the values of E_0, E_1, and E_2 and multiplying by $M_0^{\frac{3}{2}}$, our equation becomes

$$\left\{-D_2\frac{a+b}{2}+D_3\left[1-3\left(\frac{a+b}{2}\right)^2+ab\right]\right\}M_0$$
$$-\tfrac{1}{2}\left\{D_0+D_2\left[1-\tfrac{3}{2}\left(\frac{a+b}{2}\right)^2+\tfrac{1}{2}ab\right]\right\}M_1$$
$$+\left[D_1+D_0\frac{a+b}{2}\right]\left[-\tfrac{1}{2}M_2+\tfrac{3}{8}\frac{M_1^2}{M_0}\right]=0.$$

But

$$\frac{a+b}{2}M_0=-D_0D_1+\tfrac{1}{2}ab\,M_1,$$
$$-D_2\frac{a+b}{2}M_0=D_0D_1D_2-\tfrac{1}{2}D_2ab\,M_1,$$
$$\tfrac{1}{2}M_1=D_1D_2+D_0D_3+\frac{a+b}{2}M_2,$$
$$-\tfrac{1}{2}D_0M_1=-D_0D_1D_2-D_0^2D_3-D_0\frac{a+b}{2}M_2.$$

By substituting these, the equation becomes

$$-D_0^2D_3+D_3\left[1-3\left(\frac{a+b}{2}\right)^2+ab\right]M_0-\tfrac{1}{2}D_2\left[1-\tfrac{3}{2}\left(\frac{a+b}{2}\right)^2+\tfrac{3}{2}abM_1\right]$$
$$-\tfrac{1}{2}D_1\left[M_2-\tfrac{3}{4}\frac{M_1^2}{M_0}\right]-D_0\frac{a+b}{2}\left[\tfrac{3}{2}M_2-\tfrac{3}{8}\frac{M_1^2}{M_0}\right]=0.$$

This may easily be transformed into

$$-D_0^2D_3+D_3\left[1+D_1^2+3D\,D_1\frac{a+b}{2}+D_0^2-1\right]$$
$$-D_1D_2^2\left[1-\tfrac{3}{2}\left(\frac{a+b}{2}\right)^2+\tfrac{3}{2}ab\right]$$
$$-\tfrac{1}{2}D_1\left[D_2^2+2D_1D_3-3\frac{D_1^2D_2^2}{M_0}\right]$$
$$-D_0\frac{a+b}{2}\left[\tfrac{3}{2}D_2^2+3D_1D_3-\tfrac{3}{2}\frac{D_1^2D_2^2}{M_0}\right]=0.$$

Which reduces to

$$-\left[1+\tfrac{3}{2}\frac{a+b}{2}\frac{D_0D_1}{M_0}+\tfrac{3}{2}\frac{D_0^2-1}{M_0}\right]D_1D_2^2-\tfrac{1}{2}D_1\left[D_2^2-3\frac{D_1^2D_2^2}{M_0}\right]$$
$$-D_0\frac{a+b}{2}\left[\tfrac{3}{2}D_2^2-\tfrac{3}{2}\frac{D_1^2D_2^2}{M_0}\right]=0,$$

and thence to

$$-\left[1+\tfrac{1}{2}+\tfrac{3}{2}\frac{D_0^2-1}{D_1^2+1}-\tfrac{3}{2}\frac{D_1^2}{D_1^2+1}-\tfrac{3}{2}\frac{D_0^2}{D_1^2+1}\right]D_1D_2^2=0,$$

which is perceived to be identical.

When

$$\frac{1}{\sqrt{Q}}=E_0+E_1x+E_2x^2+\dots$$

is divided by $1 - x$ the remainder is equivalent to what $\frac{1}{\sqrt{Q}}$ becomes when x is put equal to 1. But

$$\frac{1}{\sqrt{Q}} = \sqrt{\frac{(\mathrm{a}-x)(x-\mathrm{b})}{1-x^2-H^2}},$$

consequently this remainder is

$$\pm \frac{\sqrt{(1-\mathrm{a})(1-\mathrm{b})}}{H(1)},$$

the ambiguous sign being so taken as to render the quantity positive. In like manner it is shown that the remainder of $\frac{1}{\sqrt{Q}}$ divided by $1 + x$ is

$$\pm \frac{\sqrt{(1+\mathrm{a})(1+\mathrm{b})}}{H(-1)}.$$

Then

$$\frac{d(\tilde{\omega} - \frac{1}{2}Ct)}{d\psi} = \mp \tfrac{1}{2} \frac{\sqrt{(1-\mathrm{a})(1-\mathrm{b})}}{1-x} + L_0 + L_1 x + L_2 x^2 + \ldots,$$

where the upper or lower sign is to be taken according as $H(1)$ is positive or negative. And

$$\frac{d(\tilde{\omega}' - \frac{1}{2}Ct)}{d\psi} = \mp \tfrac{1}{2} \frac{\sqrt{(1+\mathrm{a})(1+\mathrm{b})}}{1+x} + L'_0 + L'_1 x + L'_2 x^2 + \ldots,$$

where the upper or lower sign is to be taken according as $H(-1)$ is positive or negative. The expressions for the L and L', correct to quantities of the order of the fourth power of the eccentricities inclusive, are

$$2L_0 = \left[K\frac{dD_0}{dK} + D_0 + D_1 + D_2\right]E_0 + H(1)\left[E + E_1\right],$$

$$2L_1 = \left[K\frac{dD_1}{dK} - D_1 + D_2\right]E_0 + \left[K\frac{dD_0}{dK} + D_0 + D_1\right]E_1 + H(1)\,E_2,$$

$$2L_2 = \left[K\frac{dD_2}{dK} - D_1 - 2D_2\right]E_0 + \left[K\frac{dD_1}{dK} - D_1\right]E_1 + D_0E_2,$$

$$2L'_0 = \left[K\frac{dD_0}{dK} + D_0 + D_1 - D_2\right]E_0 - H(-1)\left[E_1 - E_2\right],$$

$$2L'_1 = \left[K\frac{dD_1}{dK} + D_1 + D_2\right]E_0 + \left[K\frac{dD_0}{dK} + D_0 + D_1\right]E_1 - H(-1)\,E_2,$$

$$2L'_2 = \left[K\frac{dD_2}{dK} - D_1 + 2D_2\right]E_0 + \left[K\frac{dD_1}{dK} + D_1\right]E_1 + D_0E_2.$$

By substituting the value

$$x = \frac{\mathrm{a}+\mathrm{b}}{2} - \frac{\mathrm{a}-\mathrm{b}}{2}\cos\psi,$$

and putting

$$N_0 = L_0 + L_1\frac{a+b}{2} + L_2\left[\tfrac{3}{2}\left(\frac{a+b}{2}\right)^2 - \tfrac{1}{2}ab\right],$$
$$N_1 = -L_1\frac{a-b}{2} - L_2\frac{a^2-b^2}{2},$$
$$N_2 = L_2\frac{(a-b)^2}{8},$$
$$N'_0 = L'_0 + L'_1\frac{a+b}{2} + L'_2\left[\tfrac{3}{2}\left(\frac{a+b}{2}\right)^2 - \tfrac{1}{2}ab\right],$$
$$N'_1 = -L'_1\frac{a-b}{2} - L'_2\frac{a^2-b^2}{2},$$
$$N'_2 = L'_2\frac{(a-b)^2}{8},$$

where we have, as has been proved above, the relation $N_0 = N_0'$, we get

$$\frac{d(\tilde{\omega} - \frac{1}{2}Ct)}{d\psi} = \mp\frac{\frac{1}{2}\sqrt{(1-a)(1-b)}}{1 - \frac{a+b}{2} + \frac{a-b}{2}\cos\psi} + N_0 + N_1\cos\psi + N_2\cos 2\psi + \ldots,$$
$$\frac{d(\tilde{\omega}' - \frac{1}{2}Ct)}{d\psi} = \mp\frac{\frac{1}{2}\sqrt{(1+a)(1+b)}}{1 + \frac{a+b}{2} - \frac{a-b}{2}\cos\psi} + N'_0 + N'_1\cos\psi + N'_2\cos 2\psi + \ldots.$$

Integrating, we have

$$\tilde{\omega} - \tfrac{1}{2}Ct = c \mp \arctan\left[\sqrt{\frac{1-a}{1-b}}\tan\frac{\psi}{2}\right] + N_0\psi + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots,$$
$$\tilde{\omega}' - \tfrac{1}{2}Ct = c' \mp \arctan\left[\sqrt{\frac{1+a}{1+b}}\tan\frac{\psi}{2}\right] + N'_0\psi + N'_1\sin\psi + \tfrac{1}{2}N'_2\sin 2\psi + \ldots.$$

The quadrant in which the arc correspondent to the tangent is to be taken is found by dividing the number of the quadrant of ψ by 2, if it is even; or by augmenting the number of the quadrant of ψ by unity, if it is odd, and then dividing by 2.

By taking the sine, we have, β being any arbitrary angle,

$$\begin{aligned}\sqrt{1-x}\,&\sin(\tilde{\omega} - \tfrac{1}{2}Ct + \beta)\\ &= \mp\sqrt{1-a}\sin\frac{\psi}{2}\cos[N_0\psi + c + \beta + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots]\\ &\quad + \sqrt{1-b}\cos\frac{\psi}{2}\sin[N_0\psi + c + \beta + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots],\end{aligned}$$

$$\begin{aligned}\sqrt{1+x}\,&\sin(\tilde{\omega}' - \tfrac{1}{2}Ct + \beta)\\ &= \mp\sqrt{1+a}\sin\frac{\psi}{2}\cos[N'_0\psi + c' + \beta + N'_1\sin\psi + \tfrac{1}{2}N'_2\sin 2\psi + \ldots]\\ &\quad + \sqrt{1+b}\cos\frac{\psi}{2}\sin[N'_0\psi + c' + \beta + N'_1\sin\psi + \tfrac{1}{2}N'_2\sin 2\psi + \ldots]\end{aligned}$$

or, as they may be written,

$$\begin{aligned}
&\sqrt{1-x}\sin(\tilde{\omega}-\tfrac12 Ct+\beta)\\
&\quad=\tfrac12[\sqrt{1-\mathrm{b}}\mp\sqrt{1-\mathrm{a}}]\sin[(N_0+\tfrac12)\psi+c+\beta+N_1\sin\psi+\tfrac12 N_2\sin 2\psi+\ldots]\\
&\quad+\tfrac12[\sqrt{1-\mathrm{b}}\pm\sqrt{1-\mathrm{a}}]\sin[(N_0-\tfrac12)\psi+c+\beta+N_1\sin\psi+\tfrac12 N_2\sin 2\psi+\ldots],\\
&\sqrt{1+x}\sin(\tilde{\omega}'-\tfrac12 Ct+\beta)\\
&\quad=\tfrac12[\sqrt{1+\mathrm{b}}\mp\sqrt{1+\mathrm{a}}]\sin[(N_0'+\tfrac12)\psi+c'+\beta+N_1'\sin\psi+\tfrac12 N_2'\sin 2\psi+\ldots]\\
&\quad+\tfrac12[\sqrt{1+\mathrm{b}}\pm\sqrt{1+\mathrm{a}}]\sin[(N_0'-\tfrac12)\psi+c'+\beta+N_1'\sin\psi+\tfrac12 N_2'\sin 2\psi+\ldots].
\end{aligned}$$

The expression for the auxiliary angle ψ in terms of the time, which has already been obtained, we will denote as follows:

$$\psi=\theta_0(t+c_0)+K_1\sin\theta_0(t+c_0)+K_2\sin 2\theta_0(t+c_0)+\ldots.$$

Substituting this for ψ in the preceding formulæ, and putting in succession

$$\beta=\tfrac12 Ct,\qquad \beta=90^\circ+\tfrac12 Ct,$$

we get

$$\begin{aligned}
\sqrt{1-x}\,{}^{\sin}_{\cos}\,\tilde{\omega}&=\tfrac12[\sqrt{1-\mathrm{b}}\mp\sqrt{1-\mathrm{a}}]\,{}^{\sin}_{\cos}[P_0+\tfrac12)\theta_0(t+c_0)+c\\
&\qquad\qquad+P_1\sin\theta_0(t+c_0)+P_2\sin 2\theta_0(t+c_0)+\ldots]\\
&+\tfrac12[\sqrt{1-\mathrm{b}}\pm\sqrt{1-\mathrm{a}}]\,{}^{\sin}_{\cos}[(P_0-\tfrac12)\theta_0(t+c_0)+c\\
&\qquad\qquad+Q_1\sin\theta_0(t+c_0)+Q_2\sin 2\theta_0(t+c_0)+\ldots],\\
\sqrt{1+x}\,{}^{\sin}_{\cos}\,\tilde{\omega}'&=\tfrac12[\sqrt{1+\mathrm{b}}\mp\sqrt{1+\mathrm{a}}]\,{}^{\sin}_{\cos}[(P_0+\tfrac12)\theta_0(t+c_0)+c'\\
&\qquad\qquad+P_1'\sin\theta_0(t+c_0)+P_2'\sin 2\theta_0(t+c_0)+\ldots]\\
&+\tfrac12\sqrt{[1+\mathrm{b}}\pm\sqrt{1+\mathrm{a}}]\,{}^{\sin}_{\cos}[(P_0-\tfrac12)\theta_0(t+c_0)+c'\\
&\qquad\qquad+Q_1'\sin\theta_0(t+c_0)+Q_2'\sin 2\theta_0(t+c_0)+\ldots].
\end{aligned}$$

Here we have put

$$P_0=N_0+\tfrac12\frac{C}{\theta_0},$$

$$\begin{aligned}
P_1&=N_1+(N_0+\tfrac12)K_1,\\
P_2&=\tfrac12[N_2+N_1K_1+2(N_0+\tfrac12)K_2],\\
Q_1&=N_1+(N_0-\tfrac12)K_1,\\
Q_2&=\tfrac12[N_2+N_1K_1+2(N_0-\tfrac12)K_2],\\
P_1'&=N_1'+(N_0+\tfrac12)K_1,\\
P_2'&=\tfrac12[N_2'+N_1'K_1+2(N_0+\tfrac12)K_2],\\
Q_1'&=N_1'+(N_0-\tfrac12)K_1,\\
Q_2'&=\tfrac12[N_2'+N_1'K_1+2(N_0-\tfrac12)K_2].
\end{aligned}$$

It is evident from the equivalent of $\sin\nu\cos\gamma$ derived from these equations that $c'=c$ or $c'=c+180^\circ$, according as

$$H(\mathrm{b})=D_0+D_1\mathrm{b}+D_2\mathrm{b}^2+D_3\mathrm{b}^3+\ldots=\pm\sqrt{1-\mathrm{b}^2}$$

is positive or negative. Hence the latter of the two equations may be written

$$
\begin{aligned}
\sqrt{1+x}\,{\textstyle{\sin \atop \cos}}\,\tilde{\varpi}' = \pm\tfrac{1}{2}[\sqrt{1+\mathfrak{b}} \mp \sqrt{1+\mathfrak{a}}]\,{\textstyle{\sin \atop \cos}}&[(P_0+\tfrac{1}{2})\theta_0(t+c_0)+c \\
&+ P_1' \sin\theta_0(t+c_0) + P_2' \sin 2\theta_0(t+c_0) + \dots] \\
\pm\tfrac{1}{2}[\sqrt{1+\mathfrak{b}} \pm \sqrt{1+\mathfrak{a}}]\,{\textstyle{\sin \atop \cos}}&[(P_0-\tfrac{1}{2})\theta_0(t+c_0)+c \\
&+ Q_1' \sin\theta_0(t+c_0) + Q_2' \sin 2\theta_0(t+c_0) + \dots]
\end{aligned}
$$

where the upper or lower of the newly introduced ambiguous signs is taken according as $H(\mathfrak{b})$ is positive or negative.

Let us put

$$
\begin{aligned}
\chi &= (P_0+\tfrac{1}{2})\theta_0(t+c_0)+c, \\
\chi' &= (P_0-\tfrac{1}{2})\theta_0(t+c_0)+c, \\
\Delta &= \tfrac{1}{2}[\sqrt{1-\mathfrak{b}} \mp \sqrt{1-\mathfrak{a}}], \\
\Delta_1 &= \tfrac{1}{2}[\sqrt{1-\mathfrak{b}} \pm \sqrt{1-\mathfrak{a}}], \\
\Delta' &= \pm\tfrac{1}{2}[\sqrt{1+\mathfrak{b}} \mp \sqrt{1+\mathfrak{a}}], \\
\Delta_1' &= \pm\tfrac{1}{2}[\sqrt{1+\mathfrak{b}} \pm \sqrt{1+\mathfrak{a}}].
\end{aligned}
$$

Then

$$
\begin{aligned}
\sqrt{1-x}\,{\textstyle{\sin \atop \cos}}\,\tilde{\varpi} = {} & [\Delta(1-\tfrac{1}{4}P_1^2)+\tfrac{1}{2}\Delta_1 Q_1]\,{\textstyle{\sin \atop \cos}}\,\chi \\
&+[\Delta_1(1-\tfrac{1}{4}Q_1^2)-\tfrac{1}{2}\Delta P_1]\,{\textstyle{\sin \atop \cos}}\,\chi' \\
&+[\tfrac{1}{2}\Delta P_1+\Delta_1(\tfrac{1}{8}Q_1^2+\tfrac{1}{2}Q_2)]\,{\textstyle{\sin \atop \cos}}\,(2\chi-\chi') \\
&+[-\tfrac{1}{2}\Delta_1 Q_1+\Delta(\tfrac{1}{8}P_1^2-\tfrac{1}{2}P_2)]\,{\textstyle{\sin \atop \cos}}\,(2\chi'-\chi) \\
&+\Delta(\tfrac{1}{8}P_1^2+\tfrac{1}{2}P_2)\,{\textstyle{\sin \atop \cos}}\,(3\chi-2\chi') \\
&+\Delta_1(\tfrac{1}{8}Q_1^2-\tfrac{1}{2}Q_2)\,{\textstyle{\sin \atop \cos}}\,(3\chi'-2\chi), \\
\sqrt{1+x}\,{\textstyle{\sin \atop \cos}}\,\tilde{\varpi}' = {} & [\Delta'(1-\tfrac{1}{4}P_1'^2)+\tfrac{1}{2}\Delta_1' Q_1']\,{\textstyle{\sin \atop \cos}}\,\chi \\
&+[\Delta_1'(1-\tfrac{1}{4}Q_1'^2)-\tfrac{1}{2}\Delta' P_1']\,{\textstyle{\sin \atop \cos}}\,\chi' \\
&+[\tfrac{1}{2}\Delta' P_1'+\Delta_1'(\tfrac{1}{8}Q_1'^2+\tfrac{1}{2}Q_2')]\,{\textstyle{\sin \atop \cos}}\,(2\chi-\chi') \\
&+[-\tfrac{1}{2}\Delta_1' Q_1'+\Delta'(\tfrac{1}{8}P_1'^2-\tfrac{1}{2}P_2')]\,{\textstyle{\sin \atop \cos}}\,(2\chi'-\chi) \\
&+\Delta'(\tfrac{1}{8}P_1'^2+\tfrac{1}{2}P_2')\,{\textstyle{\sin \atop \cos}}\,(3\chi-2\chi') \\
&+\Delta_1'(\tfrac{1}{8}Q_1'^2-\tfrac{1}{2}Q_2')\,{\textstyle{\sin \atop \cos}}\,(3\chi'-2\chi).
\end{aligned}
$$

It is evident that $e\,{\textstyle{\sin \atop \cos}}\,\tilde{\varpi}$ and $e'\,{\textstyle{\sin \atop \cos}}\,\tilde{\varpi}$ can be expressed in series of the same form.

In applying to Jupiter and Saturn these equations, it is found that by varying the value of K,

$$K\frac{dD_0}{dK} = +\,0.0101629, \qquad K\frac{dD_1}{dK} = +\,0.0009178, \qquad K\frac{dD_2}{dK} = -\,0.0050568.$$

Also

$$\log\left[-\sqrt{(1-\mathrm{a})(1-\mathrm{b})}\right] = 9.9574334n, \quad \log\sqrt{(1+\mathrm{a})(1+\mathrm{b})} = 9.4074864,$$

$$\begin{aligned} L_0 &= +\,0.1672972, & L_0' &= +\,0.1655301,\\ L_1 &= +\,0.0028107, & L_1' &= -\,0.0012760,\\ L_2 &= -\,0.0000071, & L_2' &= -\,0.0000250.\end{aligned}$$

Whence

$$\begin{aligned} N_0 &= +\,0.1667632, & N_0' &- +\,0.1667632,\\ N_1 &= -\,0.0021649, & N_1' &= +\,0.0009746,\\ N_2 &= -\,0.0000021, & N_2' &= -\,0.0000074.\end{aligned}$$

Also

$$\begin{aligned} P_0 &= +\,0.6837293, & P_1' &= -\,786''.82,\\ P_1 &= -\,1434''.41, & P_2' &= +\quad 2''.54,\\ P_2 &= +\quad 5''.41, & Q_1' &= +\,694''.76,\\ Q_1 &= +\quad 47''.17, & Q_2' &= -\quad 3''.50,\\ Q_2 &= -\quad 0''.63. \end{aligned}$$

$$\begin{aligned} \log\varDelta &= 9.5750158, & \log\varDelta' &= 9.8634412n,\\ \log\varDelta_1 &= 0.0101623, & \log\varDelta_1' &= 9.7217366,\\ (P_0+\tfrac{1}{2})\,\theta_0 &= 22''.55981, & (P_0-\tfrac{1}{2})\,\theta_0 &= 3''.50156.\end{aligned}$$

$$\begin{aligned}\sqrt{1-x}\,\frac{\sin}{\cos}\,\tilde\omega = &+0.3759635\frac{\sin}{\cos}\chi \qquad + 1.0249824\frac{\sin}{\cos}\chi'\\ &-0.0013085\frac{\sin}{\cos}(2\chi-\chi') - 0.0001196\frac{\sin}{\cos}(2\chi'-\chi)\\ &+0.0000072\frac{\sin}{\cos}(3\chi-2\chi') + 0.0000016\frac{\sin}{\cos}(3\chi'-2\chi),\end{aligned}$$

$$\begin{aligned}\sqrt{1+x}\,\frac{\sin}{\cos}\,\tilde\omega' = &-0.7293089\frac{\sin}{\cos}\chi \qquad + 0.5255160\frac{\sin}{\cos}\chi'\\ &+0.0013889\frac{\sin}{\cos}(2\chi-\chi') - 0.0008842\frac{\sin}{\cos}(2\chi'-\chi)\\ &-0.0000058\frac{\sin}{\cos}(3\chi-2\chi') + 0.0000052\frac{\sin}{\cos}(3\chi'-2\chi).\end{aligned}$$

The value of c is found to be

$$c = 340^\circ\ 8'\ 50''.26.$$

Hence the expressions for the two arguments are

$$\begin{aligned}\chi &= 308^\circ\ 13'\ 15''.13 + 22''.55981t,\\ \chi' &= \ \ 31^\circ\ \ 4'\ \ 5''.98 + \ 3''.50156t.\end{aligned}$$

The following expressions for e and e' were obtained:

$$\frac{e}{\sqrt{1-x}} = [8.6282138]\sqrt{(1-[6.5410419]\cos\psi)},$$

$$\frac{e'}{\sqrt{1+x}} = [8.8231642]\sqrt{(1+[6.9312571]\cos\psi)},$$

$$\frac{e}{\sqrt{1-x}} = [8.6282135]\,\{1-[6.24001]\cos(\chi-\chi') + [3.7900]\cos 2(\chi-\chi')\},$$

$$\frac{e'}{\sqrt{1+x}} = [8.8231648]\,\{1+[6.63023]\cos(\chi-\chi') - [4.1982]\cos 2(\chi-\chi')\}.$$

By means of these we can pass to the expressions for the following functions

$$
\begin{aligned}
e \tfrac{\sin}{\cos} \tilde{\omega} = &+ 0.01596822 \tfrac{\sin}{\cos} \chi &&+ 0.04354278 \tfrac{\sin}{\cos} \chi' \\
&- 0.00005696 \tfrac{\sin}{\cos} (2\chi - \chi') &&- 0.00000886 \tfrac{\sin}{\cos} (2\chi' - \chi) \\
&+ 0.00000031 \tfrac{\sin}{\cos} (3\chi - 2\chi') &&+ 0.00000009 \tfrac{\sin}{\cos} (3\chi' - 2\chi), \\
e' \tfrac{\sin}{\cos} \tilde{\omega}' = &- 0.04852990 \tfrac{\sin}{\cos} \chi &&+ 0.03496407 \tfrac{\sin}{\cos} \chi' \\
&+ 0.00008205 \tfrac{\sin}{\cos} (2\chi - \chi') &&- 0.00005134 \tfrac{\sin}{\cos} (2\chi' - \chi) \\
&- 0.00000033 \tfrac{\sin}{\cos} (3\chi - 2\chi') &&+ 0.00000031 \tfrac{\sin}{\cos} (3\chi' - 2\chi).
\end{aligned}
$$

It will be observed that these expressions are as convergent as could be wished. The form of these integrals being discovered, another and more direct method of arriving at them is suggested. The coefficients being assumed as indeterminate as well as the rates of movement of the two arguments together with the constants which complete the values of the latter, the expressions could be substituted in the differential equations, and thus would arise twelve equations of condition, which along with the values of the four variables at the origin of time would determine the sixteen unknowns involved. But on trial it seems this way of proceeding would necessitate as long computations as the method we have followed.

In conclusion, it may be observed that, if terms arising from the squares and higher powers of the masses were taken into consideration, the form of this investigation would not thereby be changed; the only effect produced would be that the values of the various constants involved would receive slight modifications.

MEMOIR No. 48

DETERMINATION OF THE INEQUALITIES OF THE MOON'S MOTION WHICH ARE PRODUCED BY THE FIGURE OF THE EARTH

A SUPPLEMENT TO DELAUNEY'S LUNAR THEORY

(Astronomical Papers of the American Ephemeris, Vol. III, pp. 201-344, 1884.)

PREFACE.

Since its appearance, DELAUNAY's Theory of the Moon's motion has, very generally, been regarded by astronomers as a great advance on any previous treatment of the subject. Especially is it admired on account of the orderly and methodical arrangement of the matter and the elegant processes employed in its elaboration. Hence it has been regretted that this theory was left unfinished at DELAUNAY's death. The solar perturbations were quite fully treated, but the subordinate portions of the subject were either incomplete or untouched. At the time it was hinted that some of the French astronomers would undertake to fill up these gaps. But more than ten years have elapsed and nothing has appeared except a very elaborate treatment of a long-period inequality due to the action of Mars, by M. GOGOU.

Under these circumstances it has seemed that it might be permitted to me to take up a portion of the subject untouched by DELAUNAY, viz, the perturbations which the moon undergoes on account of the figure of the earth.

The sensible character of these inequalities was discovered by LAPLACE; but he and his immediate successors contented themselves with determining the coefficients of two periodic terms; one of the fourth order in the longitude, the other in the latitude and of the third order, whose periods depend on the position of the moon's node with reference to the equinox. The most elaborate treatment of this subject, we at present have, is by HANSEN. It appears in his memoir entitled "*Darlegung, &c.*"* The coefficients of about twenty terms are computed, and all that can be of utility for the formation of the most exact tables are supposed to be there contained. But these coefficients appear in the work only as numbers; hence it is impossible to see to what cause they owe their magnitude. Moreover, no regard has been paid to the algebraic order of magnitude in retaining or rejecting terms. Thus it will be seen that, in this portion of the subject, we have nothing to compare with DELAUNAY's splendid treatment of the solar perturbations.

The problem, then, which I propose to solve in this memoir is to determine, in a literal form, all the inequalities of the moon which arise from the figure of the earth, to the same degree of algebraical approximation as DELAUNAY has adopted in determining the solar perturbations, viz, to terms of the seventh order inclusive. It might be thought that, as the numerical factors in this case are much smaller than in the case of the solar perturbations, this is a degree of approximation greater than is needed for practical purposes. However, we note that the largest term of the seventh order which appears in our expressions has the value $0''.0291$; and that three or four of our coefficients are probably in error more than $0''.01$ from neglected terms. Hence, it has appeared better to retain seventh-order terms and submit to the inconvenience

* Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, Band XI, ss 273–322.

of determining a multitude of terms which are practically insignificant. The methods of proceeding and the notation, with one exception, which will be pointed out hereafter, are, as nearly as I can imagine, those which DELAUNAY would have employed. Hence I hope there is no impropriety in entitling this memoir a supplement to DELAUNAY'S Theory.

The term which ought to be added to the perturbative function R, in order to take into account the figure of the earth, is, employing the usual notation,

$$\frac{3}{2}\frac{M+m}{M}\left(C-\frac{A+B}{2}\right)\frac{\frac{1}{3}-\sin^2\delta}{r^3}.$$

To follow DELAUNAY'S method, we must, in the first place, substitute for r and δ their values in terms of the six quantities a, e, γ, l, g and h, deduced from the formulæ of elliptic motion. This gives rise to an expression, which, written to the degree of approximation we require, contains twenty-seven periodic terms. At this point DELAUNAY would undoubtedly have made in the expression the transformations which he has called "Operations," and numbered from 1 onwards, and then retained only such terms as were necessary for his purpose. But, in this way, it is often difficult to see what terms may be neglected. In some of the coefficients of R the approximation must be pushed to terms of the eighth order, in others to the ninth or tenth, and in one even to the eleventh order. Thus, employing DELAUNAY'S values of the solar perturbations of the three co-ordinates of the moon, given at the end of his second volume, I have preferred to make use of TAYLOR'S theorem extended to three variables. Here it is found unnecessary to go beyond terms of two dimensions. This is the only deviation I have permitted myself from what would probably have been DELAUNAY'S method of proceeding.

In this way an expression for R is obtained which contains one hundred and twenty-two periodic terms. Following DELAUNAY'S process, these terms must, in succession, be removed from R by a series of operations. The number of these operations is one hundred and three. These substitutions must also be made in the values of the three co-ordinates of the moon as they are affected by solar perturbation, and which DELAUNAY has given at the end of his second volume. When the new terms, which thus arise, are reduced to their simplest expression, it is found that the perturbations of the moon's longitude, due to the figure of the earth, contain one hundred and sixty-five periodic terms, the perturbations of the latitude two hundred and nine terms, and the perturbations of the horizontal parallax five terms. In the last I have adopted the same degree of approximation as DELAUNAY. The motions of the perigee and node, due to the figure of the earth, are then determined, and correct to quantities of the eighth order inclusive.

It remains now to turn these literal expressions into numerical formulæ. For this purpose we need the value of the constant factor

$$\frac{3}{2}\frac{M+m}{M}\left(C-\frac{A+B}{2}\right),$$

which multiplies the whole of each expression. Here three independent sources offer

themselves, from which this value may be obtained. First, it may be obtained from a discussion of the observations of the moon, which method has been followed by HANSEN. But, to do this properly, requires an exact knowledge of certain inequalities produced by the direct and indirect action of the planets, and having nearly the same periods as the terms arising from the figure of the earth. This also is a portion of the lunar theory left untouched by DELAUNAY. The value of the constant, derived in this way, would not have a high degree of precision. In the second place, the value may be obtained from geodetic measures. Lastly, which appears to me the preferable method, and is the one I have adopted, it may be obtained from the measures of the intensity of gravity made at stations supposed to lie on a level surface.

When the subject is treated in the most general manner possible, we get a system of four equations, from which, if we eliminate three unknown quantities denoting the co-ordinates of the point in space, we have an equation giving the value of the intensity of gravity in terms of the geographical longitude and latitude of the station. These equations involve the potential of the attraction of all points of the earth's mass. In the ignorance in which we are of the peculiar figure of the earth's bounding surface and of its interior constitution as regards density, the triple integration, which this potential demands, is accomplished by the aid of an infinite series consisting of spherical or harmonic functions. Each of these functions contains a certain number of constants not necessarily having any dependence on each other. Hence the series will contain a certain number of constants, which is greater or less according as the series is extended to a greater or less length. Having observations of the intensity of gravity at a certain number of stations, the series could be given such a length as to contain as many constants as there were stations. The observations would then determine all these constants; and the formula, thus obtained for g, would, on the substitution in it, of the appropriate longitude and latitude, exactly regive the observed value. But, in this way, the elimination would be an almost impracticable task, and we are obliged to be content with a far less number of disposable constants. The expression, which I have employed as the value of the triple integral involved in the potential, contains twenty constants; and as we have more equations than unknowns, the method of least squares is used to obtain a solution.

These unknown constants are really the values of the series of definite integrals, contained in the general formula

$$\int\int\int \rho\, x^i y^j z^k \, dx\, dy\, dz,$$

where ρ denotes the density of the earth at the point xyz, and i, j, and k positive integers, and the integration must be extended to all points of the earth's mass. Hence it will be seen that the constant factor, whose value we need in getting the perturbations of the moon produced by the figure of the earth, may be regarded as being one of these constants. Thus, in conducting the elimination of the unknowns in the normal equations the method of least squares furnishes, we get rid of the unknowns whose values are unnecessary to our purpose, and obtain a single equation affording the value of the special constant we need.

In obtaining formulæ for representing the intensity of gravity over the earth's

surface, previous investigators have confined themselves to two, or, at the most, to three disposable constants. And THOMSON and TAIT* have discouraged the adoption of more complex expressions. In their view the outstanding deviations are very local in their character, and, consequently, in order to their being wiped out, the addition of spherical functions of a high order would be required. But such a conclusion should be drawn from the results of an actual investigation. On account of the extremely unequal distribution over the earth's surface of the stations, at which, up to the present time, gravity has been measured, it certainly appears possible that very different values of the particular constant, necessary in the determination of the lunar perturbations arising from the figure of the earth, might be obtained, according as more or less of disposable constants were admitted into the formula. As matter of fact, the use of twenty constants has given nearly the same result as the use of two. This coincidence, however, must be regarded as accidental.

Although the values of the eighteen additional constants, obtained in my investigation, have extremely small weight, and are sure to be overturned when determinations of gravity shall have been made in regions at present uncovered by stations, I have, nevertheless, written down the resulting formula for the length of the second's pendulum. It is of interest as showing that the determinations we have at present, can be as well represented by a formula containing quite large terms involving the longitude of the station, as by a formula which is a function of the latitude only.

By direction of Professor NEWCOMB, Mr. HENRY MEIER has made a duplicate of the somewhat tedious computations of Chapter V.

* Treatise on Natural Philosophy, Part II, p. 365.

LUNAR INEQUALITIES PRODUCED BY THE FIGURE OF THE EARTH.

CHAPTER I.

DETERMINATION AND DEVELOPMENT IN PERIODIC SERIES OF THE PART OF THE PERTURBATIVE FUNCTION WHICH DEPENDS ON THE FIGURE OF THE EARTH.

If M and m denote severally the masses of the earth and moon, and dM and dm their elements, and Δ the distance between the latter, the potential function Ω, for the interaction of these bodies, will be determined by the equation

$$\Omega = \int\int \frac{d\mathrm{M}\, dm}{\Delta},$$

the summation being extended so as to include every pair of elements of the two masses. Again, if Ω be so expressed as to involve the rectangular co-ordinates X, Y, and Z of the center of gravity of the earth, and also those of the center of gravity of the moon, viz, ξ, η, and ζ, the differential equations of motion of these centers of gravity will be, for the earth,

$$\mathrm{M}\frac{d^2\mathrm{X}}{dt^2} = \frac{d\Omega}{d\mathrm{X}},$$
$$\mathrm{M}\frac{d^2\mathrm{Y}}{dt^2} = \frac{d\Omega}{d\mathrm{Y}},$$
$$\mathrm{M}\frac{d^2\mathrm{Z}}{dt^2} = \frac{d\Omega}{d\mathrm{Z}},$$

and for the moon,

$$m\frac{d^2\xi}{dt^2} = \frac{d\Omega}{d\xi},$$
$$m\frac{d^2\eta}{dt^2} = \frac{d\Omega}{d\eta},$$
$$m\frac{d^2\zeta}{dt^2} = \frac{d\Omega}{d\zeta}.$$

Let x, y, and z denote the rectangular co-ordinates of the center of gravity of the moon relative to the center of gravity of the earth, so that we have

$$\xi - \mathrm{X} = x,$$
$$\eta - \mathrm{Y} = y,$$
$$\zeta - \mathrm{Z} = z.$$

If Ω is now so expressed as to involve the variables x, y, and z, we shall have

$$\frac{d\Omega}{d\xi} = -\frac{d\Omega}{dX} = \frac{d\Omega}{dx},$$
$$\frac{d\Omega}{d\eta} = -\frac{d\Omega}{dY} = \frac{d\Omega}{dy},$$
$$\frac{d\Omega}{d\zeta} = -\frac{d\Omega}{dZ} = \frac{d\Omega}{dz},$$

And, consequently,

$$\frac{d^2x}{dt^2} = \frac{1}{m}\frac{d\Omega}{d\xi} - \frac{1}{M}\frac{d\Omega}{dX} = \frac{M+m}{Mm}\frac{d\Omega}{dx},$$
$$\frac{d^2y}{dt^2} = \frac{1}{m}\frac{d\Omega}{d\eta} - \frac{1}{M}\frac{d\Omega}{dY} = \frac{M+m}{Mm}\frac{d\Omega}{dy},$$
$$\frac{d^2z}{dt} = \frac{1}{m}\frac{d\Omega}{d\zeta} - \frac{1}{M}\frac{d\Omega}{dZ} = \frac{M+m}{Mm}\frac{d\Omega}{dz},$$

If we suppose the co-ordinates of dM, relative to the center of gravity of M, are denoted by X', Y' and Z', and those of dm, relative to the center of gravity of m, by ξ', η', and ζ', we shall have

$$\Omega = \int\int \frac{dM\, dm}{[(x+\xi'-X')^2 + (y+\eta'-Y')^2 + (z+\zeta'-Z')^2]^{\frac{1}{2}}}$$

But, as we do not propose to take into account the inequalities arising from the figure of the moon, we shall assume that the bounding surface of this body is spherical, and that its mass is either homogeneous or that the density of the element dm is a function of its distance from the center of the bounding sphere. In this case, the integration involved in the last expression, relative to dm, can be accomplished; and the known result is

$$\Omega = m\int \frac{dM}{[(x-X')^2 + (y-Y')^2 + (z-Z')^2]^{\frac{1}{2}}}.$$

If we write

$$r^2 = x^2 + y^2 + z^2, \qquad r'^2 = X'^2 + Y'^2 + Z'^2,$$

we have

$$\left[(x-X')^2 + (y-Y')^2 + (z-Z')^2\right]^{-\frac{1}{2}} = \frac{1}{r}\left[1 - 2\frac{xX' + yY' + zZ'}{r^2} + \frac{r'^2}{r^2}\right]^{-\frac{1}{2}}.$$

The second term of the radical of the right-hand member of this equation is a quantity of the order of the ratio of the dimensions of the terrestrial spheroid to the radius of the lunar orbit, and the third term is of the order of the square of this ratio. Hence, developing, in a series, this radical, and agreeing to neglect terms of the order of the cube and higher powers of the mentioned ratio, and remembering that, by the properties of the center of gravity, we have the equations

$$\int X'dM = 0, \qquad \int Y'dM = 0, \qquad \int Z'dM = 0,$$

we may write

$$\Omega = m\int \frac{dM}{r}\left[1 - \frac{r'^2}{2r^2} + \frac{3}{2}\frac{(xX' + yY' + zZ')^2}{r^4}\right].$$

Here it may be noted, that when we suppose the bounding surface of the earth, as well as the surfaces of equal density, to be of revolution about a common axis, and that these surfaces are cut by the plane of the equator into symmetrical halves, we shall have the equation

$$\int f(\mathrm{X}', \mathrm{Y}', \mathrm{Z}')\, d\mathrm{M} = 0,$$

where f denotes any rational integral function composed of terms of odd dimensions with reference to X', Y' and Z'. In this case, therefore, all terms of odd orders vanish from the development of Ω in series, and the expression, given above, for this quantity, is correct to terms of the fourth order. We also assume that the earth rotates about the axis of maximum moment, and consequently that the two other principal axes lie in the plane of the equator. Hence α denoting the moon's right ascension and δ its declination, we may have

$$\begin{aligned} x &= r \cos\delta \cos\alpha, \\ y &= r \cos\delta \sin\alpha, \\ z &= r \sin\delta. \end{aligned}$$

Moreover, ω denoting the right ascension of the point of the heavens which is met by the prolongation of the axis of x', we may assume a system of co-ordinates x', y' and z' referred to the principal axes of the earth, such that

$$\begin{aligned} \mathrm{X}' &= x' \cos\omega + y' \sin\omega, \\ \mathrm{Y}' &= x' \sin\omega - y' \cos\omega, \\ \mathrm{Z}' &= z'. \end{aligned}$$

We shall then have

$$\int x'y'\,d\mathrm{M} = 0, \qquad \int x'z'\,d\mathrm{M} = 0, \qquad \int y'z'\,d\mathrm{M} = 0,$$

and, in the usual notation,

$$\int (y'^2 + z'^2)\, d\mathrm{M} = \mathrm{A}, \qquad \int (x'^2 + z'^2)\, d\mathrm{M} = \mathrm{B}, \qquad \int (x'^2 + y'^2)\, d\mathrm{M} = \mathrm{C}.$$

On making these substitutions in the expression for Ω, we obtain

$$\Omega = m\left[\frac{\mathrm{M}}{r} + \frac{3}{2}\left(\mathrm{C} - \frac{\mathrm{A}+\mathrm{B}}{2}\right)\frac{\frac{1}{3} - \sin^2\delta}{r^3} - \frac{3}{4}(\mathrm{A} - \mathrm{B})\frac{\cos^2\delta}{r^3}\cos(2\alpha - 2\omega)\right].$$

But it is evident the last term of this expression can give rise, in the lunar co-ordinates, only to inequalities whose period is about half a day, at least when quantities of the order of the square of this disturbing force are neglected, as we propose to do. Moreover, as the motion of the arguments of these inequalities is about fifty-five times more rapid than that of the moon in its orbit, integration will cause the coefficients of these terms in the expression of the forces to be divided by the large divisor 55^2. In addition, the difference $\mathrm{A} - \mathrm{B}$ is known to be very small in comparison with the difference $\mathrm{C} - \frac{\mathrm{A}+\mathrm{B}}{2}$. Hence we shall reject the term in question.

Thus the term which ought to be added to the perturbative function R, on account of the figure of the earth, is

$$R = \frac{3}{2}\frac{M+m}{M}\left(C - \frac{A+B}{2}\right)\frac{\frac{1}{3} - \sin^2\delta}{r^3}.$$

In order to follow DELAUNAY's method, we must, in the first place, substitute for r and δ their values in terms of the six quantities a, e, γ, l, g, h deduced from the formulæ of elliptic motion. Let V denote the longitude of the moon measured from a fixed equinox upon the corresponding fixed ecliptic of a certain date, as, for instance, of the beginning of 1850. Let U denote the corresponding latitude, and ε the obliquity of the equator of date upon the mentioned ecliptic, and ψ the luni-solar precession from 1850.0 to date. Then we shall have

$$\sin\delta = \cos\varepsilon \sin U + \sin\varepsilon \cos U \sin(V + \psi).$$

Denoting, with DELAUNAY, the angular distance of the moon from its ascending node by ν, and the inclination of its orbit to the plane of the mentioned ecliptic by i, we shall have the equations

$$\begin{aligned} \sin U &= \sin i \sin\nu, \\ \cos U \cos(V - h) &= \cos\nu, \\ \cos U \sin(V - h) &= \cos i \sin\nu. \end{aligned}$$

Substituting these values in the expression for $\sin\delta$, and adopting DELAUNAY's γ in place of i, we get

$$\sin\delta = 2\gamma(1-\gamma^2)^{\frac{1}{2}}\cos\varepsilon\sin\nu + (1-\gamma^2)\sin\varepsilon\sin(\psi + h + \nu) + \gamma^2\sin\varepsilon\sin(\psi + h - \nu).$$

Squaring this expression we have

$$\begin{aligned} \frac{1}{3} - \sin^2\delta = &\left(\frac{1}{3} - 2\gamma^2 + 2\gamma^4\right)\left(1 - \frac{3}{2}\sin^2\varepsilon\right) \\ &+ 2\gamma^2(1-\gamma^2)\left(1 - \frac{3}{2}\sin^2\varepsilon\right)\cos 2\nu \\ &- \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\sin 2\varepsilon\cos(\psi + h) \\ &+ \gamma(1-\gamma^2)^{\frac{3}{2}}\sin 2\varepsilon\cos(\psi + h + 2\nu) \\ &- \gamma^3(1-\gamma^2)^{\frac{1}{2}}\sin 2\varepsilon\cos(\psi + h - 2\nu) \\ &+ \gamma^2(1-\gamma^2)\sin^2\varepsilon\cos(2\psi + 2h) \\ &+ \frac{1}{2}(1-\gamma^2)^2\sin^2\varepsilon\cos(2\psi + 2h + 2\nu) \\ &+ \frac{1}{2}\gamma^4\sin^2\varepsilon\cos(2\psi + 2h - 2\nu). \end{aligned}$$

For brevity's sake we will put

$$\beta_1 = \frac{3}{2}\frac{1}{M}\left(C - \frac{A+B}{2}\right)\left(1 - \frac{3}{2}\sin^2\varepsilon\right),$$

$$\beta_2 = \frac{3}{2}\frac{1}{M}\left(C - \frac{A+B}{2}\right)\sin 2\varepsilon,$$

$$\beta_3 = \frac{3}{2}\frac{1}{M}\left(C - \frac{A+B}{2}\right)\sin^2\varepsilon.$$

With DELAUNAY we will denote $M + m$ by μ, and for $(1 - \gamma^2)^{\frac{1}{2}}$ and $(1 - \gamma^2)^{\frac{3}{2}}$ will substitute their expressions in powers of γ, neglecting all powers above the fifth. Then the perturbative function has the following expression:

$$
\begin{aligned}
R = {} & \frac{\beta_1\mu}{a^3}\left[\frac{1}{3} - 2\gamma^2 + 2\gamma^4\right]\frac{a^3}{r^3} \\
& + 2\frac{\beta_1\mu}{a^3}[\gamma^2 - \gamma^4]\frac{a^3}{r^3}\cos 2v \\
& - \frac{\beta_2\mu}{a^3}\left[\gamma - \frac{5}{2}\gamma^3 + \frac{7}{8}\gamma^5\right]\frac{a^3}{r^3}\cos(\psi + h) \\
& + \frac{\beta_2\mu}{a^3}\left[\gamma - \frac{3}{2}\gamma^3 + \frac{3}{8}\gamma^5\right]\frac{a^3}{r^3}\cos(\psi + h + 2v) \\
& - \frac{\beta_2\mu}{a^3}\left[\gamma^3 - \frac{1}{2}\gamma^5\right]\frac{a^3}{r^3}\cos(\psi + h - 2v) \\
& + \frac{\beta_3\mu}{a^3}[\gamma^2 - \gamma^4]\frac{a^3}{r^3}\cos(2\psi + 2h) \\
& + \frac{\beta_3\mu}{a^3}\left[\frac{1}{2} - \gamma^2 + \frac{1}{2}\gamma^4\right]\frac{a^3}{r^3}\cos(2\psi + 2h + 2v) \\
& + \frac{1}{2}\frac{\beta_3\mu}{a^3}\gamma^4\frac{a^3}{r^3}\cos(2\psi + 2h - 2v).
\end{aligned}
$$

DELAUNAY has determined all the lunar inequalities arising from the solar action to the seventh order inclusive, without exception, with some of the eighth and ninth orders, calling e, γ, and m quantities of the first order of smallness. The large numerical factors, which the terms of high orders often have, renders necessary this extended degree of approximation. Although this circumstance does not exist in the class of inequalities we propose to determine, and, hence, we might content ourselves with a lower degree of approximation, yet, for the sake of uniformity, I have set the seventh order as the degree of the terms we shall stop with. However, no terms involving the squares or products of the three quantities β_1, β_2, and β_3 will be considered. I have made no investigation of the order of these terms, but presume that they are of no significance. This convention demands we should neglect in ε the lunar nutation of the obliquity. This quantity contains also a very small term proportional to t^2, which we shall neglect. Hence, we regard ε as a constant. The three quantities β_1, β_2, β_3 are then constants, and it is evident that, with our conventions, the three portions of R, severally factored by them, give rise to three classes of inequalities in the moon's

co-ordinates, which are entirely independent of each other; the first having arguments independent of ψ, the second having arguments involving the simple multiple of ψ, and the third having arguments involving 2ψ. Our convention would demand that in integrating we should neglect the motion of ψ, but I have written in the coefficients the few terms which thus arise, calling this motion divided by the moon's mean motion a quantity of the fifth order.

If D denote the equatorial radius of the earth, it is evident that the order of the constants β_1, β_2, β_3 ought to be regarded as the same as that of the quantity

$$\frac{D^2}{a^2}\,\frac{C - \frac{1}{2}(A + B)}{MD^2}.$$

The first factor of this is nearly equivalent to $\left(\frac{1}{60}\right)^2 = \frac{1}{3600}$, and may be regarded as of the third order. The second is the order of the compression of the earth, which is nearly $\frac{1}{300}$, and may be called of the second order. Hence β_1, β_2, β_3 and R are quantities of the fifth order.

In order to get all the inequalities belonging to the first seven orders, it is necessary to push the development of R in general to terms of the eighth order, and to include besides all ninth order terms whose arguments do not contain l, and all tenth order terms whose arguments contain neither l nor l'. In addition to this one argument has presented itself, viz, $\psi + 2h + g - h' - g'$, whose movement is a quantity of the order of $\frac{n'^3}{n^3}$; hence its coefficient must be determined correctly to terms of the eleventh order inclusive.

It will be observed that the elliptic expansion of R depends on that of the two functions $\frac{a^3}{r^3}$ and $\frac{a^3}{r^3}\cos(\alpha - 2v)$, where α denotes any arbitrary angle, here to be put, in succession, equal to 0, $-(\psi + h)$, $\psi + h$, $-(2\psi+2h)$, $2\psi+2h$. The development of these functions has been given by DELAUNAY.* They are as follows:

$$\begin{aligned}\frac{a^3}{r^3} = 1 &+ \frac{3}{2}e^2 + \frac{15}{8}e^4 \\ &+ \left(3e + \frac{27}{8}e^3 + \frac{261}{64}e^5\right)\cos l \\ &+ \left(\frac{9}{2}e^2 + \frac{7}{2}e^4\right)\cos 2l \\ &+ \left(\frac{53}{8}e^3 + \frac{393}{128}e^5\right)\cos 3l \\ &+ \frac{77}{8}e^4\cos 4l \\ &+ \frac{1773}{128}e^5\cos 5l;\end{aligned}$$

*Mémoires de l'Académie des Sciences de Paris. Tom. XXVIII, pp. 27–28. They may be found developed two orders further in a Memoir by Professor CAYLEY, Mem. Roy. Astr. Soc., Vol. XXIX.

$$\frac{a^3}{r^3}\cos(\alpha-2\nu)=\left(1-\frac{5}{2}e^2+\frac{13}{16}e^4\right)\cos(\alpha-2g-2l)$$
$$+\left(\frac{7}{2}e-\frac{123}{16}e^3+\frac{489}{128}e^5\right)\cos(\alpha-2g-3l)$$
$$-\left(\frac{1}{2}e-\frac{1}{16}e^3+\frac{5}{384}e^5\right)\cos(\alpha-2g-l)$$
$$+\left(\frac{17}{2}e^2-\frac{115}{16}e^4\right)\cos(\alpha-2g-4l)$$
$$+(0\,e^2+0\,e^4)\cos(\alpha-2g)$$
$$+\left(\frac{845}{48}e^3-\frac{32525}{768}e^5\right)\cos(\alpha-2g-5l)$$
$$+\left(\frac{1}{48}e^3+\frac{11}{768}e^5\right)\cos(\alpha-2g+l)$$
$$+\frac{533}{16}e^4\cos(\alpha-2g-6l)$$
$$+\frac{1}{24}e^4\cos(\alpha-2g+2l)$$
$$+\frac{228347}{3840}e^5\cos(\alpha-2g-7l)$$
$$+\frac{81}{1280}e^5\cos(\alpha-2g+3l).$$

When these two expressions are substituted in the last expression for R, and only the terms which can be useful to us preserved, we get

$$\mathrm{R}=\frac{\beta_1\mu}{a^3}\left[\frac{1}{3}-2\gamma^2+\frac{1}{2}e^2+2\gamma^4-3\gamma^2e^2+\frac{5}{8}e^4\right]$$
$$+\frac{\beta_1\mu}{a^3}\left[e-6\gamma^2e+\frac{9}{8}e^3\right]\cos l$$
$$+\frac{3}{2}\frac{\beta_1\mu}{a^3}e^2\cos 2l$$
$$+\frac{53}{24}\frac{\beta_1\mu}{a^3}e^3\cos 3l$$
$$+2\frac{\beta_1\mu}{a^3}\gamma^2\cos(2g+2l)$$
$$+7\frac{\beta_1\mu}{a_3}\gamma^2e\cos(2g+3l)$$
$$-\frac{\beta_1\mu}{a^3}\gamma^2e\cos(2g+l)$$
$$+\frac{\beta_2\mu}{a^3}\left[\gamma-\frac{3}{2}\gamma^3-\frac{5}{2}\gamma e^2\right]\cos(\psi+h+2g+2l)$$
$$+\frac{7}{2}\frac{\beta_2\mu}{a^3}\gamma e\cos(\psi+h+2g+3l)$$
$$+\frac{17}{2}\frac{\beta_2\mu}{a^3}\gamma e^2\cos(\psi+h+2g+4l)$$
$$-\frac{\beta_2\mu}{a^3}\left[\frac{1}{2}\gamma e-\frac{3}{4}\gamma^3e-\frac{1}{16}\gamma e^3\right]\cos(\psi+h+2g+l)$$

$$-\frac{\beta_3\mu}{a^3}\left[\gamma-\frac{5}{2}\gamma^3+\frac{3}{2}\gamma e^2+\frac{7}{8}\gamma^5-\frac{15}{4}\gamma^3e^2+\frac{15}{8}\gamma e^4\right]\cos(\psi+h)$$

$$-\frac{\beta_3\mu}{a^3}\left[\frac{3}{2}\gamma e-\frac{15}{4}\gamma^3 e+\frac{27}{16}\gamma e^3\right]\cos(\psi+h+l)$$

$$-\frac{\beta_3\mu}{a^3}\left[\frac{9}{4}\gamma e^2-\frac{45}{8}\gamma^3e^2+\frac{7}{4}\gamma e^4\right]\cos(\psi+h+2l)$$

$$-\frac{\beta_3\mu}{a^3}\left[\frac{3}{2}\gamma e-\frac{15}{4}\gamma^3 e+\frac{27}{16}\gamma e^3\right]\cos(\psi+h-l)$$

$$-\frac{\beta_3\mu}{a^3}\left[\frac{9}{4}\gamma e^2-\frac{45}{8}\gamma^3e^2+\frac{7}{4}\gamma e^4\right]\cos(\psi+h-2l)$$

$$-\frac{\beta_3\mu}{a^3}\gamma^3\cos(\psi+h-2g-2l)$$

$$+\frac{1}{2}\frac{\beta_3\mu}{a^3}\gamma^3 e\cos(\psi+h-2g-l)$$

$$+\frac{\beta_3\mu}{a^3}\left[\frac{1}{2}-\gamma^2-\frac{5}{4}e^2\right]\cos(2\psi+2h+2g+2l)$$

$$+\frac{\beta_3\mu}{a^3}\left[\frac{7}{4}e-\frac{7}{2}\gamma^2e-\frac{123}{32}e^3\right]\cos(2\psi+2h+2g+3l)$$

$$+\frac{17}{4}\frac{\beta_3\mu}{a^3}e^2\cos(2\psi+2h+2g+4l)$$

$$+\frac{845}{96}\frac{\beta_3\mu}{a^3}e^3\cos(2\psi+2h+2g+5l)$$

$$-\frac{\beta_3\mu}{a^3}\left[\frac{1}{4}e-\frac{1}{2}\gamma^2e-\frac{1}{32}e^3\right]\cos(2\psi+2h+2g+l)$$

$$+\frac{1}{96}\frac{\beta_3\mu}{a^3}e^3\cos(2\psi+2h+2g-l)$$

$$+\frac{\beta_3\mu}{a^3}\left[\gamma^2-\gamma^4+\frac{3}{2}\gamma^2e^2\right]\cos(2\psi+2h)$$

$$+\frac{3}{2}\frac{\beta_3\mu}{a^3}\gamma^2e\cos(2\psi+2h+l)$$

$$+\frac{3}{2}\frac{\beta_3\mu}{a^3}\gamma^2e\cos(2\psi+2h-l).$$

The readiest method of getting the additional terms of R, which are produced by the action of the sun, appears to be the employment of TAYLOR's theorem. Let us call the preceding value of R, R_0, and put r for $\frac{a}{r}$. Let δr, δV, δU denote the increments of r, V and U due to the solar action. Then we shall have

$$\begin{aligned}R=R_0&+\frac{dR_0}{dr}\delta r+\frac{dR_0}{dV}\delta V+\frac{dR_0}{dU}\delta U\\&+\frac{1}{2}\frac{d^2R_0}{dr^2}\delta r^2+\frac{1}{2}\frac{d^2R_0}{dV^2}\delta V^2+\frac{1}{2}\frac{d^2R_0}{dU^2}\delta U^2\\&+\frac{d^2R_0}{drdV}\delta r\delta V+\frac{d^2R_0}{drdU}\delta r\delta U+\frac{d^2R_0}{dVdU}\delta V\delta U.\end{aligned}$$

As δr, δV and δU are quantities of the second order, the three terms of the preceding equation, which involve these quantities to one dimension, give rise, in R, to terms

which, at the lowest, are of the seventh order. And the six terms, which involve their squares and products of two dimensions, give rise to terms which, at the lowest, are of the ninth order. The terms involving products of δr, δV and δU of three dimensions are, at lowest, of the eleventh order; and hence need only be considered for the term whose argument is $\psi + 2h + g - h' - g'$. But the coefficient of this term has the quantity $\frac{a}{a'}$ as a factor; and, on inspection, it will be found that the terms of δr, δV and δU, which have this factor, are, at lowest, of the third order. Thus the terms of products of δr, δV and δU, of three dimensions, having $\frac{a}{a'}$ as a factor, are, at lowest, of the seventh order, and, consequently, can give rise in R to terms which are, at lowest, of the twelfth order. Hence the preceding expression for R, as written, has all the extension necessary for our purpose.

We will consider the terms of this expression in their order.

I. We have, omitting all terms of orders higher than the eighth,

$$\begin{aligned}
\frac{dR_0}{dr} &= \frac{\beta_1\mu}{a^3}[1 - 6\gamma^2]\frac{a^3}{r^3} \\
&+ 6\frac{\beta_1\mu}{a^3}\gamma^2\frac{a^3}{r^3}\cos 2v \\
&- \frac{\beta_2\mu}{a^3}\left[3\gamma - \frac{15}{2}\gamma^3\right]\frac{a^3}{r^3}\cos(\psi + h) \\
&+ \frac{\beta_2\mu}{a^3}\left[3\gamma - \frac{9}{2}\gamma^3\right]\frac{a^3}{r^3}\cos(\psi + h + 2v) \\
&- 3\frac{\beta_2\mu}{a^3}\gamma^3\frac{a^3}{r^3}\cos(\psi + h - 2v) \\
&+ 3\frac{\beta_3\mu}{a^3}\gamma^2\frac{a^3}{r^3}\cos(2\psi + 2h) \\
&+ \frac{\beta_3\mu}{a^3}\left[\frac{3}{2} - 3\gamma^2\right]\frac{a^3}{r^3}\cos(2\psi + 2h + 2v).
\end{aligned}$$

The development of this function depends on those of the functions $\frac{a^3}{r^3}$ and $\frac{a^3}{r^3}\cos(\alpha - 2v)$. We have, to the degree of accuracy necessary,

$$\frac{a^3}{r^3} = 1 + \frac{1}{2}e^2 + 2e\cos l + \frac{5}{2}e^2\cos 2l,$$

$$\frac{a^3}{r^3}\cos(\alpha - 2v) = \left(1 - \frac{7}{2}e^2\right)\cos(\alpha - 2g - 2l) - e\cos(\alpha - 2g - l)$$
$$+ 3e\cos(\alpha - 2g - 3l).$$

Substituting these values, and preserving only the terms that can be useful,

$$\frac{dR_0}{dr} = \frac{\beta_1\mu}{a^3}\left[1 - 6\gamma^2 + \frac{1}{2}e^2\right] \qquad (1)$$

$$+ 2\frac{\beta_1\mu}{a^3}e\cos l \qquad (2)$$

$$+ \frac{5}{2}\frac{\beta_1\mu}{a^3}e^2\cos 2l \qquad (3)$$

13 $$+ 6\frac{\beta_1\mu}{a^3}\gamma^2\cos(2g + 2l) \qquad (4)$$

$$+3\frac{\beta_3\mu}{a^3}\gamma\cos(\psi+h+2g+2l) \qquad (5)$$

$$-3\frac{\beta_3\mu}{a^3}\gamma e\cos(\psi+h+2g+l) \qquad (6)$$

$$-\frac{\beta_3\mu}{a^3}\left[3\gamma-\frac{15}{2}\gamma^3+\frac{3}{2}\gamma e^2\right]\cos(\psi+h) \qquad (7)$$

$$-3\frac{\beta_3\mu}{a^3}\gamma e\cos(\psi+h+l) \qquad (8)$$

$$-3\frac{\beta_3\mu}{a^3}\gamma e\cos(\psi+h-l) \qquad (9)$$

$$+\frac{\beta_3\mu}{a^3}\left[\frac{3}{2}-3\gamma^2-\frac{21}{4}e^2\right]\cos(2\psi+2h+2g+2l) \qquad (10)$$

$$+\frac{9}{2}\frac{\beta_3\mu}{a^3}e\cos(2\psi+2h+2g+3l) \qquad (11)$$

$$-\frac{3}{2}\frac{\beta_3\mu}{a^3}e\cos(2\psi+2h+2g+l) \qquad (12)$$

$$+3\frac{\beta_3\mu}{a^3}\gamma^2\cos(2\psi+2h). \qquad (13)$$

We take now from DELAUNAY* the value of δr. The following is a statement of the rule which must guide us in the selection of terms to be retained. First, all terms of the second and third orders without exception; second, all terms of the fourth order whose arguments do not contain l or contain $2l$, or which, wanting l', contain $\pm l$ or $\pm 3l$; third, all terms of the fifth order, which, not containing l' in their arguments, do not contain l or contain $\pm 2l$. The term in R having the argument $\psi+2h+g-h'-g'$ needing special consideration, it is readily seen that the factor from $\frac{dR_0}{dr}$, producing it, is of the sixth order; hence it will be sufficient to take into account terms of δr to the fifth order; and, to this degree of approximation, it is found that only one term of δr can produce it, viz, that having the argument $h+g+l-h'-g'$.

$$\delta r=\left(\frac{1}{6}+\frac{1}{4}e'^2\right)m^2-\frac{179}{288}m^4-\frac{97}{48}m^5 \qquad (1)$$

$$-\frac{3}{2}e'm^2\cos l' \qquad (2)$$

$$-\frac{9}{4}e'^2m^2\cos 2l' \qquad (3)$$

$$-\left(\frac{7}{12}em^2+\frac{285}{64}em^3\right)\cos l \qquad (4)$$

$$+\frac{21}{8}ee'm\cos(l-l') \qquad (5)$$

$$-\frac{21}{8}ee'm\cos(l+l') \qquad (6)$$

$$-\left(\frac{5}{6}e^2m^2+\frac{735}{64}e^2m^3\right)\cos 2l \qquad (7)$$

$$+\frac{21}{4}e^2e'm\cos(2l-l') \qquad (8)$$

*Mémoires de l'Académie des Sciences de Paris, Tom. XXIX, pp. 914–924.

$$-\frac{21}{4}e^2e'm\cos(2l+l') \tag{9}$$

$$\left(5\gamma^2e^2-\frac{135}{8}\gamma^2e^2m-2\gamma^2m^2+3\gamma^2m^3\right)\cos(2g+2l) \tag{10}$$

$$-\left(\frac{5}{2}\gamma^2e-\frac{135}{16}\gamma^2em\right)\cos(2g+l) \tag{11}$$

$$+\left[\frac{15}{4}e^2m+\left(1-2\gamma^2+\frac{189}{16}e^2-\frac{5}{2}e'^2\right)m^2+\frac{19}{6}m^3+\frac{131}{18}m^4\right]$$
$$\times\cos(2h+2g+2l-2h'-2g'-2l') \tag{12}$$

$$+\left(\frac{35}{4}e^2e'm+\frac{7}{2}e'm^2+\frac{157}{8}e'm^3\right)\cos(2h+2g+2l-2h'-2g'-3l') \tag{13}$$

$$+\frac{17}{2}e'^2m^2\cos(2h+2g+2l-2h'-2g'-4l') \tag{14}$$

$$-\left(\frac{15}{4}e^2e'm+\frac{1}{2}e'm^2+\frac{91}{24}e'm^3\right)\cos(2h+2g+2l-2h'-2g'-l') \tag{15}$$

$$-\left(\frac{45}{16}e^2e'^2m+\frac{3}{4}e'^2m^2\right)\cos(2h+2g+2l-2h'-2g') \tag{16}$$

$$+\frac{33}{16}em^2\cos(2h+2g+3l-2h'-2g'-2l') \tag{17}$$

$$+\left(\frac{15}{8}em+\frac{187}{32}em^2\right)\cos(2h+2g+l-2h'-2g'-2l') \tag{18}$$

$$+\frac{35}{8}ee'm\cos(2h+2g+l-2h'-2g'-3l') \tag{19}$$

$$-\frac{15}{8}ee'm\cos(2h+2g+l-2h'-2g'-l') \tag{20}$$

$$-\frac{45}{32}ee'^2m\cos(2h+2g+l-2h'-2g') \tag{21}$$

$$-\frac{15}{4}e^2m^2\cos(2h+2g-2h'-2g'-2l') \tag{22}$$

$$-3\gamma^2m^2\cos(2h-2h'-2g'-2l') \tag{23}$$

$$+\frac{225}{64}e^2m^2\cos(4h+4g+2l-4h'-4g'-4l') \tag{24}$$

$$-\frac{15}{16}m\frac{a}{a'}\cos(h+g+l-h'-g'-l') \tag{25}$$

$$+\left(\frac{5}{4}e'-\frac{45}{8}e'm\right)\frac{a}{a'}\cos(h+g+l-h'-g') \tag{26}$$

$$-\frac{15}{8}em\frac{a}{a'}\cos(h+g+2l-h'-g'-l') \tag{27}$$

$$+\left(\frac{5}{2}ee'-\frac{45}{2}ee'm\right)\frac{a}{a'}\cos(h+g+2l-h'-g'). \tag{28}$$

The terms of R, which arise from the multiplication of the two factors just given, and which ought, in accordance with our conventions, to be retained, will be found in the expression given hereafter, with the indication of the terms of the two factors from

whose combination they arise; thus the terms, underscored in the manner [I. 11 .... 7], result from the multiplication of the term numbered (11) in $\frac{dR_0}{dr}$ by the term numbered (7) in δr. The same indication will be given in all the multiplications which follow.

II. We have

$$\frac{dR_0}{dV} = \frac{dR_0}{d\psi} = -\frac{\beta_3\mu}{a^3}\gamma \sin(\psi + h + 2g + 2l) \quad (1)$$

$$+\frac{1}{2}\frac{\beta_3\mu}{a^3}\gamma e \sin(\psi + h + 2g + l) \quad (2)$$

$$+\frac{\beta_3\mu}{a^3}\gamma \sin(\psi + h) \quad (3)$$

$$+\frac{3}{2}\frac{\beta_3\mu}{a^3}\gamma e \sin(\psi + h + l) \quad (4)$$

$$+\frac{3}{2}\frac{\beta_3\mu}{a^3}\gamma e \sin(\psi + h - l) \quad (5)$$

$$-\frac{\beta_3\mu}{a^3}\left[1 - 2\gamma^2 - \frac{5}{2}e^2\right]\sin(2\psi + 2h + 2g + 2l) \quad (6)$$

$$-\frac{7}{2}\frac{\beta_3\mu}{a^3}e \sin(2\psi + 2h + 2g + 3l) \quad (7)$$

$$+\frac{1}{2}\frac{\beta_3\mu}{a^3}e \sin(2\psi + 2h + 2g + l) \quad (8)$$

$$-2\frac{\beta_3\mu}{a^3}\gamma^2 \sin(2\psi + 2h). \quad (9)$$

We take now from DELAUNAY* the value of δV. The rules, which guide us in the selection of terms to be retained, are as follows. First, all terms of the second and third orders without exception; second, all terms of the fourth order whose arguments contain $\pm 2l$, or which, wanting l', contain $0l$, $\pm l$ or $\pm 3l$; third, all terms of the fifth order, which, not containing l' in their arguments, contain $\pm 2l$. And, in order to get the coefficient of $\cos(\psi + 2h + g - h' - g')$ to the required degree of approximation, it is found necessary to include in the coefficient of $\sin(h + g - h' - g')$ the term of the fifth order.

$$\delta V = -3e'm \sin l' \quad (1)$$

$$-\frac{9}{4}e'^2 m \sin 2l' \quad (2)$$

$$+\frac{21}{4}ee'm \sin(l - l') \quad (3)$$

$$-\frac{21}{4}ee'm \sin(l + l') \quad (4)$$

$$+\left[-\frac{5}{4}\gamma^2e^2 + \frac{135}{32}\gamma^2e^2m - \frac{7}{16}e^2m^2 - \frac{2595}{256}e^2m^3\right]\sin 2l \quad (5)$$

$$+\frac{105}{16}e^2e'm \sin(2l - l') \quad (6)$$

$$-\frac{105}{16}e^2e'm \sin(2l + l') \quad (7)$$

* Tom. II, pp. 803–861.

$$+\left[-\frac{25}{4}\gamma^2e^2+\frac{675}{32}\gamma^2e^2m+\frac{11}{4}\gamma^2m^2-\frac{231}{64}\gamma^2m^3\right]\sin(2g+2l) \tag{8}$$

$$-\frac{3}{4}\gamma^2e'm\sin(2g+2l-l') \tag{9}$$

$$+\frac{3}{4}\gamma^2e'm\sin(2g+2l+l') \tag{10}$$

$$+\left[-5\gamma^2e+\frac{135}{8}\gamma^2em\right]\sin(2g+l) \tag{11}$$

$$+\frac{5}{4}\gamma^2e^2\sin 2g \tag{12}$$

$$+\left[\left(-\frac{3}{4}\gamma^2+\frac{75}{16}e^2\right)m+\left(\frac{11}{8}-\frac{47}{16}\gamma^2+\frac{1101}{64}e^2-\frac{55}{16}e'^2\right)m^2+\frac{59}{12}m^3+\frac{893}{72}m^4\right]$$
$$\times\sin(2h+2g+2l-2h'-2g'-2l') \tag{13}$$

$$+\left[\left(-\frac{7}{4}\gamma^2e'+\frac{175}{16}e^2e'\right)m+\frac{77}{16}e'm^2+\frac{479}{16}e'm^3\right]\sin(2h+2g+2l-2h'-2g'-3l') \tag{14}$$

$$+\frac{187}{16}e'^2m^2\sin(2h+2g+2l-2h'-2g'-4l') \tag{15}$$

$$+\left[\left(\frac{3}{4}\gamma^2e'-\frac{75}{16}e^2e'\right)m-\frac{11}{16}e'm^2-\frac{257}{48}e'm^3\right]\sin(2h+2g+2l-2h'-2g'-l') \tag{16}$$

$$+\left[\left(\frac{9}{16}\gamma^2e'^2-\frac{225}{64}e^2e'^2\right)m-\frac{33}{32}e'^2m^3\right]\sin(2h+2g+2l-2h'-2g') \tag{17}$$

$$+\frac{17}{8}em^2\sin(2h+2g+3l-2h'-2g'-2l') \tag{18}$$

$$+\left[\frac{15}{4}em+\frac{263}{16}em^2\right]\sin(2h+2g+l-2h'-2g'-2l') \tag{19}$$

$$+\frac{35}{4}ee'm\sin(2h+2g+l-2h'-2g'-3l') \tag{20}$$

$$-\frac{15}{4}ee'm\sin(2h+2g+l-2h'-2g'-l') \tag{21}$$

$$-\frac{45}{16}ee'^2m\sin(2h+2g+l-2h'-2g') \tag{22}$$

$$+\frac{45}{16}e^2m\sin(2h+2g-2h'-2g'-2l') \tag{23}$$

$$+\frac{9}{4}\gamma^2m\sin(2h-2h'-2g'-2l') \tag{24}$$

$$+\frac{1125}{256}e^2m^2\sin(4h+4g+2l-4h'-4g'-4l') \tag{25}$$

$$-\frac{9}{64}\gamma^2m^2\sin(4h+2g+2l-4h'-4g'-4l') \tag{26}$$

$$-\frac{15}{8}m\frac{a}{a'}\sin(h+g+l-h'-g'-l') \tag{27}$$

$$+\left[\frac{5}{2}e'-\frac{45}{4}e'm\right]\frac{a}{a'}\sin(h+g+l-h'-g') \tag{28}$$

$$-\frac{75}{32}em\frac{a}{a'}\sin(h+g+2l-h'-g'-l') \tag{29}$$

$$+\left[\frac{25}{8}ee'-\frac{225}{16}ee'm\right]\frac{a}{a'}\sin(h+g+2l-h'-g') \tag{30}$$

$$+\left[\frac{25}{8}ee'-\frac{495}{16}ee'm\right]\frac{a}{a'}\sin(h+g-h'-g'). \tag{31}$$

III. We have

$$\frac{d\mathrm{R}_0}{d\mathrm{U}} = -\frac{\beta_1\mu}{a_3}\frac{a^3}{r^3}\sin 2\mathrm{U} - \frac{\beta_2\mu}{a^3}\frac{a^3}{r^3}\cos 2\mathrm{U}\sin(\mathrm{V}+\psi) - \frac{1}{2}\frac{\beta_3\mu}{a^3}\frac{a^3}{r^3}\sin 2\mathrm{U}\cos 2(\mathrm{V}+\psi).$$

In developing this expression it is found that it is unnecessary to retain any powers of γ above the second. To this degree of approximation

$$\begin{aligned}
\sin 2\mathrm{U} &= 4\gamma\sin\nu,\\
\cos 2\mathrm{U} &= 1 - 4\gamma^2 + 4\gamma^2\cos 2\nu,\\
\sin(\mathrm{V}+\psi) &= \sin(\psi+h+\nu) + \frac{1}{2}\gamma^2\sin(\psi+h-\nu) - \frac{1}{2}\gamma^2\sin(\psi+h+3\nu),\\
\cos 2(\mathrm{V}+\psi) &= \cos(2\psi+2h+2\nu) + \gamma^2\cos(2\psi+2h) - \gamma^2\cos(2\psi+2h+4\nu).
\end{aligned}$$

On making these substitutions we get

$$\begin{aligned}
\frac{d\mathrm{R}_0}{d\mathrm{U}} = &- 4\frac{\beta_1\mu}{a^3}\gamma\frac{a^3}{r^3}\sin\nu\\
&- \frac{\beta_2\mu}{a^3}[1-4\gamma^2]\frac{a^3}{r^3}\sin(\psi+h+\nu)\\
&- \frac{5}{2}\frac{\beta_2\mu}{a^3}\gamma^2\frac{a^3}{r^3}\sin(\psi+h-\nu)\\
&- \frac{3}{2}\frac{\beta_2\mu}{a^3}\gamma^2\frac{a^3}{r^3}\sin(\psi+h+3\nu)\\
&- \frac{\beta_3\mu}{a^3}\gamma\frac{a^3}{r^3}\sin(2\psi+2h+3\nu)\\
&+ \frac{\beta_3\mu}{a^3}\gamma\frac{a^3}{r^3}\sin(2\psi+2h+\nu).
\end{aligned}$$

The principal term of this expression depends on the expansion of $\frac{a^3}{r^3}\sin(\alpha+\nu)$. Preserving only the terms which can be useful, we have

$$\begin{aligned}
\frac{a^3}{r^3}\sin(\alpha+\nu) = &\left(1+\frac{1}{2}e^2\right)\sin(\alpha+g+l)\\
&+ \left(\frac{5}{2}e - \frac{1}{8}e^3\right)\sin(\alpha+g+2l)\\
&+ \frac{1}{2}e\sin(\alpha+g)\\
&+ \frac{5}{8}e^2\sin(\alpha+g-l).
\end{aligned}$$

In the remaining terms it will suffice to put $\frac{a^3}{r^3} = 1$, and $\nu = g + l$. Then preserving only the terms which can be of use, we have

$$\begin{aligned}
\frac{d\mathrm{R}_0}{d\mathrm{U}} = &- 4\frac{\beta_1\mu}{a^3}\gamma\sin(g+l) && (1)\\
&- \frac{\beta_2\mu}{a^3}\left[1 - 4\gamma^2 + \frac{1}{2}e^2\right]\sin(\psi+h+g+l) && (2)\\
&- \frac{5}{2}\frac{\beta_2\mu}{a^3}e\sin(\psi+h+g+2l) && (3)
\end{aligned}$$

$$-\frac{1}{2}\frac{\beta_2\mu}{a^3}e\sin(\psi+h+g) \tag{4}$$

$$-\frac{5}{8}\frac{\beta_2\mu}{a^3}e^2\sin(\psi+h+g-l) \tag{5}$$

$$-\frac{5}{2}\frac{\beta_2\mu}{a^3}\gamma^2\sin(\psi+h-g-l) \tag{6}$$

$$-\frac{\beta_3\mu}{a^3}\gamma\sin(2\psi+2h+3g+3l) \tag{7}$$

$$+\frac{\beta_3\mu}{a^3}\gamma\sin(2\psi+2h+g+l). \tag{8}$$

We take from DELAUNAY* the value of δU. The following rules guide us in selecting the terms of δU to be retained. First, all terms of the second and third orders without exception; second, all terms of the fourth order, which have $\pm l$ in their arguments, or which, being free from l', contain $0l$, $\pm 2l$ or $\pm 3l$; third, all terms of the fifth order, whose arguments, being free from l', contain $\pm l$. In addition, in order to have the coefficient of $\cos(\psi+2h+g-h'-g')$ correct to the proposed degree of accuracy, it is necessary to include in the coefficient of $\sin(h-h'-g')$ the term of the fifth order, and in the coefficient of $\sin(h-l-h'-g')$ the term of the sixth order.

$$\delta\text{U} = \left(\frac{3}{4}\gamma e'm + \frac{9}{32}\gamma e'm^2\right)\sin(g+l-l') \tag{1}$$

$$+\frac{9}{16}\gamma e'^2m\sin(g+l-2l') \tag{2}$$

$$-\left(\frac{3}{4}\gamma e'm + \frac{69}{32}\gamma e'm^2\right)\sin(g+l+l') \tag{3}$$

$$-\frac{9}{16}\gamma e'^2m\sin(g+l+2l') \tag{4}$$

$$-\frac{1}{2}\gamma em^2\sin(g+2l) \tag{5}$$

$$+\left(-5\gamma^3e + \frac{5}{4}\gamma e^3 + \frac{189}{32}\gamma em^2\right)\sin g \tag{6}$$

$$+\left(-\frac{5}{4}\gamma e^2 - 10\gamma^3e^2 + \frac{77}{48}\gamma e^4 + \frac{135}{32}\gamma e^2m + \frac{2025}{256}\gamma e^2m^2\right)\sin(g-l) \tag{7}$$

$$-\frac{5}{4}\gamma e^3\sin(g-2l) \tag{8}$$

$$-5\gamma^3e\sin(3g+2l) \tag{9}$$

$$+\frac{5}{2}\gamma^3e^2\sin(3g+l) \tag{10}$$

$$+\frac{11}{8}\gamma m^2\sin(2h+3g+3l-2h'-2g'-2l') \tag{11}$$

$$+\frac{15}{4}\gamma em\sin(2h+3g+2l-2h'-2g'-2l') \tag{12}$$

$$-\frac{15}{32}\gamma e^2m\sin(2h+3g+l-2h'-2g'-2l') \tag{13}$$

* Tom. II, pp. 869–913.

$$+\left[\left(\frac{3}{4}\gamma+\frac{9}{8}\gamma^3+\frac{27}{16}\gamma e^2-\frac{15}{8}\gamma e'^2\right)m+\frac{25}{16}\gamma m^2+\frac{2957}{768}\gamma m^3\right]\sin(2h+g+l-2h'-2g'-2l') \qquad (14)$$

$$+\left(\frac{7}{4}\gamma e'm+\frac{255}{32}\gamma e'm^2\right)\sin(2h+g+l-2h'-2g'-3l') \qquad (15)$$

$$+\frac{51}{16}\gamma e'^2m\sin(2h+g+l-2h'-2g'-4l') \qquad (16)$$

$$-\left(\frac{3}{4}\gamma e'm+\frac{115}{32}\gamma e'm^2\right)\sin(2h+g+l-2h'-2g'-l') \qquad (17)$$

$$-\left(\frac{9}{16}\gamma e'^2m+\frac{57}{128}\gamma e'^2m^2\right)\sin(2h+g+l-2h'-2g') \qquad (18)$$

$$+\frac{3}{4}\gamma em\sin(2h+g+2l-2h'-2g'-2l') \qquad (19)$$

$$+3\gamma em\sin(2h+g-2h'-2g'-2l') \qquad (20)$$

$$+\frac{147}{32}\gamma e^2m\sin(2h+g-l-2h'-2g'-2l') \qquad (21)$$

$$+\frac{15}{8}\gamma^3m\sin(2h-g-l-2h'-2g'-2l') \qquad (22)$$

$$+\frac{5}{2}\gamma e'\frac{a}{a'}\sin(h+2g+2l-h'-g') \qquad (23)$$

$$-\frac{5}{8}\gamma ee'\frac{a}{a'}\sin(h+2g+l-h'-g') \qquad (24)$$

$$+\left(\frac{5}{2}\gamma e'-\frac{45}{4}\gamma e'm\right)\frac{a}{a'}\sin(h-h'-g') \qquad (25)$$

$$+\frac{55}{24}\gamma ee'\frac{a}{a'}\sin(h+l-h'-g') \qquad (26)$$

$$+\left(\frac{25}{8}\gamma ee'-\frac{955}{16}\gamma ee'm\right)\frac{a}{a'}\sin(h-l-h'-g'). \qquad (27)$$

IV. In obtaining the term factored by $(\delta r)^2$, it will be sufficient to take

$$\begin{aligned}\delta r=&\frac{1}{6}m^2\\&+\left[\frac{15}{4}e^2m+m^2+\frac{19}{6}m^3\right]\cos(2h+2g+2l-2h'-2g'-2l')\\&+\frac{33}{16}em^2\cos(2h+2g+3l-2h'-2g'-2l')\\&+\left[\frac{15}{8}em+\frac{187}{32}em^2\right]\cos(2h+2g+l-2h'-2g'-2l').\end{aligned}$$

Squaring, and preserving only the terms we need,

$$(\delta r)^2=\frac{225}{128}e^2m^2+\frac{3765}{256}e^2m^3+\frac{19}{36}m^4+\frac{19}{6}m^5 \qquad (1)$$

$$+\frac{15}{8}em^3\cos l \qquad (2)$$

$$+\frac{495}{128}e^2m^3\cos 2l \qquad (3)$$

$$+\frac{1}{3}m^4\cos(2h+2g+2l-2h'-2g'-2l') \quad (4)$$

$$+\frac{5}{8}em^3\cos(2h+2g+l-2h'-2g'-2l') \quad (5)$$

$$+\frac{225}{128}e^2m^3\cos(4h+4g+2l-4h'-4g'-4l'). \quad (6)$$

The value of the other factor, omitting two terms, of the sixth order, with the arguments $\psi+h+2g+2l$ and $2\psi+2h+2g+3l$, because they contribute nothing to the sought product, is

$$\frac{1}{2}\frac{d^2R_0}{dr^2}=\frac{\beta_1\mu}{a^3} \quad (1)$$

$$+\frac{\beta_1\mu}{a^3}e\cos l \quad (2)$$

$$-3\frac{\beta_2\mu}{a^3}\gamma\cos(\psi+h) \quad (3)$$

$$+\frac{3}{2}\frac{\beta_3\mu}{a^3}\cos(2\psi+2h+2g+2l) \quad (4)$$

$$-\frac{9}{4}\frac{\beta_3\mu}{a^3}e\cos(2\psi+2h+2g+l). \quad (5)$$

V. The value of the first factor of the term multiplied by $(\delta V)^2$, omitting two terms of the sixth order with the arguments $\psi+h+2g+2l$ and $2\psi+2h+2g+3l$, because they contribute nothing to the sought product, is

$$\frac{1}{2}\frac{d^2R_0}{dV^2}=\frac{1}{2}\frac{d^2R_0}{d\psi^2}=\frac{1}{2}\frac{\beta_2\mu}{a^3}\gamma\cos(\psi+h) \quad (1)$$

$$-\frac{\beta_3\mu}{a^3}\cos(2\psi+2h+2g+2l) \quad (2)$$

$$+\frac{1}{2}\frac{\beta_3\mu}{a^3}e\cos(2\psi+2h+2g+l). \quad (3)$$

In order to obtain the value of $(\delta V)^2$ it will be sufficient to take

$$\begin{aligned}\delta V = & -3e'm\sin l' \\ & -\frac{9}{4}e'^2m\sin 2l' \\ & +\frac{11}{8}m^2\sin(2h+2g+2l-2h'-2g'-2l') \\ & -\frac{11}{16}e'm^2\sin(2h+2g+2l-2h'-2g'-l') \\ & +\frac{17}{8}em^2\sin(2h+2g+3l-2h'-2g'-2l') \\ & +\frac{15}{4}em\sin(2h+2g+l-2h'-2g'-2l') \\ & +\frac{45}{16}e^2m\sin(2h+2g-2h'-2g'-2l') \\ & +\frac{9}{4}\gamma^2m\sin(2h-2h'-2g'-2l').\end{aligned}$$

Squaring, and preserving only the terms we need,

$$(\delta V)^2 = \frac{225}{32} e^2m^2 + \frac{9}{2} e'^2m^2 + \frac{121}{128} m^4 \qquad (1)$$

$$+ \frac{165}{32} em^3 \cos l \qquad (2)$$

$$+ \frac{1515}{128} e^2m^3 \cos 2l \qquad (3)$$

$$+ \frac{99}{32} \gamma^2m^3 \cos (2g + 2l) \qquad (4)$$

$$- \frac{33}{8} e'm^3 \cos (2h + 2g + 2l - 2h' - 2g' - 3l') \qquad (5)$$

$$+ \frac{33}{8} e'm^3 \cos (2h + 2g + 2l - 2h' - 2g' - l') \qquad (6)$$

$$+ \frac{33}{32} e'^2m^3 \cos (2h + 2g + 2l - 2h' - 2g'). \qquad (7)$$

VI. We have, rigorously,

$$\frac{1}{2}\frac{d^2R_0}{dU^2} = -\frac{\beta_1\mu}{a^3}\frac{a^3}{r^3}\cos 2U + \frac{\beta_2\mu}{a^3}\frac{a^3}{r^3}\sin 2U \sin (V + \psi) - \frac{1}{2}\frac{\beta_3\mu}{a^3}\frac{a^3}{r^3}\cos 2U \cos 2(V + \psi).$$

Omitting four terms, of the sixth order, whose arguments are l, $\psi + h + 2g + 2l$, $2\psi + 2h + 2g + 3l$, and $2\psi + 2h + 2g + l$, because they contribute nothing to the sought product, the sufficiently approximate value of this factor is

$$\frac{1}{2}\frac{d^2R_0}{dU^2} = -\frac{\beta_1\mu}{a^3} \qquad (1)$$

$$+ 2\frac{\beta_2\mu}{a^3}\gamma \cos (\psi + h) \qquad (2)$$

$$- \frac{1}{2}\frac{\beta_3\mu}{a^3} \cos (2\psi + 2h + 2g + 2l). \qquad (3)$$

In obtaining the value of $(\delta U)^2$, it will be sufficient to put

$$\delta U = \frac{11}{8}\gamma m^2 \sin (2h + 3g + 3l - 2h' - 2g' - 2l')$$

$$+ \left[\frac{3}{4}\gamma m + \frac{25}{16}\gamma m^2\right] \sin (2h + g + l - 2h' - 2g' - 2l').$$

Squaring, and preserving only the terms we need,

$$(\delta U)^2 = \frac{9}{32}\gamma^2m^2 + \frac{75}{64}\gamma^2m^3 \qquad (1)$$

$$+ \frac{33}{32}\gamma^2m^3 \cos (2g + 2l) \qquad (2)$$

$$- \frac{9}{32}\gamma^2m^2 \cos (4h + 2g + 2l - 4h' - 4g' - 4l'). \qquad (3)$$

VII. Omitting three terms of the sixth order, whose arguments are $\psi + h$,

$\psi + h + 2g + 2l$ and $2\psi + 2h + 2g + 3l$, because they contribute nothing to the sought product, we have

$$\frac{d^2R_0}{d\mathrm{r}d\mathrm{V}} = \frac{d^2R_0}{d\mathrm{r}d\psi} = -3\frac{\beta_3\mu}{a^3}\sin(2\psi + 2h + 2g + 2l) \quad (1)$$

$$+3\frac{\beta_3\mu}{a^3}e\sin(2\psi + 2h + 2g + l). \quad (2)$$

In deriving the product $\delta\mathrm{r}\delta\mathrm{V}$, it is sufficient to take

$$\begin{aligned}\delta\mathrm{r} = &\frac{1}{6}m^2\\ &+ m^2\cos(2h + 2g + 2l - 2h' - 2g' - 2l')\\ &-\frac{1}{2}e'm^2\cos(2h + 2g + 2l - 2h' - 2g' - l')\\ &+\frac{33}{16}em^2\cos(2h + 2g + 3l - 2h' - 2g' - 2l')\\ &+\frac{15}{8}em\cos(2h + 2g + l - 2h' - 2g' - 2l'),\end{aligned}$$

and

$$\begin{aligned}\delta\mathrm{V} = &-3e'm\sin l'\\ &-\frac{9}{4}e'^2m\sin 2l'\\ &+\frac{11}{8}m^2\sin(2h + 2g + 2l - 2h' - 2g' - 2l')\\ &+\frac{17}{8}em^2\sin(2h + 2g + 3l - 2h' - 2g' - 2l')\\ &+\frac{15}{4}em\sin(2h + 2g + l - 2h' - 2g' - 2l')\\ &+\frac{45}{16}e^2m\sin(2h + 2g - 2h' - 2g' - 2l')\\ &+\frac{9}{4}\gamma^2m\sin(2h - 2h' - 2g' - 2l').\end{aligned}$$

And, preserving only such terms as we need, the product is

$$\begin{aligned}\delta\mathrm{r}\delta\mathrm{V} = &-\frac{75}{128}em^3\sin l &(1)\\ &-\frac{195}{64}e^2m^3\sin 2l &(2)\\ &-\frac{9}{8}\gamma^2m^3\sin(2g + 2l) &(3)\\ &+\frac{11}{48}m^4\sin(2h + 2g + 2l - 2h' - 2g' - 2l') &(4)\\ &+\frac{3}{2}e'm^3\sin(2h + 2g + 2l - 2h' - 2g' - 3l') &(5)\\ &-\frac{3}{2}e'm^3\sin(2h + 2g + 2l - 2h' - 2g' - l') &(6)\\ &-\frac{3}{8}e'^2m^3\sin(2h + 2g + 2l - 2h' - 2g') &(7)\\ &+\frac{225}{64}e^2m^3\sin(4h + 4g + 2l - 4h' - 4g' - 4l'). &(8)\end{aligned}$$

VIII. Omitting two terms of the sixth order, whose arguments are $\psi + h + g + 2l$ and $2\psi + 2h + 3g + 3l$, because they contribute nothing to the sought product, we have

$$\frac{d^2R_0}{drdU} = -12\frac{\beta_1\mu}{a^3}\gamma\sin(g+l) \tag{1}$$

$$-3\frac{\beta_2\mu}{a^3}\sin(\psi+h+g+l) \tag{2}$$

$$+3\frac{\beta_3\mu}{a^3}\gamma\sin(2\psi+2h+g+l) \tag{3}$$

In obtaining the product $\delta r\delta U$, it will be sufficient to take

$$\delta r = \frac{1}{6}m^2$$
$$+\left[\frac{15}{4}e^2m + m^2 + \frac{19}{6}m^3\right]\cos(2h+2g+2l-2h'-2g'-2l')$$
$$+\frac{15}{8}em\cos(2h+2g+l-2h'-2g'-2l'),$$

and

$$\delta U = -\frac{5}{4}\gamma e^2\sin(g-l)$$
$$+\frac{11}{8}\gamma m^2\sin(2h+3g+3l-2h'-2g'-2l')$$
$$+\frac{15}{4}\gamma em\sin(2h+3g+2l-2h'-2g'-2l')$$
$$+\left[\frac{3}{4}\gamma m + \frac{25}{16}\gamma m^2\right]\sin(2h+g+l-2h'-2g'-2l')$$
$$+\frac{3}{4}\gamma em\sin(2h+g+2l-2h'-2g'-2l')$$
$$+3\gamma em\sin(2h+g-2h'-2g'-2l').$$

And, preserving only the terms we need, the product is

$$\delta r\delta U = -\left[\frac{45}{64}\gamma e^2m^2 + \frac{3}{8}\gamma m^3 + \frac{41}{32}\gamma m^4\right]\sin(g+l) \tag{1}$$

$$-\frac{175}{192}\gamma e^2m^2\sin(g-l) \tag{2}$$

$$+\frac{1}{8}\gamma m^3\sin(2h+g+l-2h'-2g'-2l'). \tag{3}$$

IX. Omitting two terms of the sixth order, whose arguments are $\psi + h + g + 2l$ and $2\psi + 2h + 3g + 3l$, because they contribute nothing to the sought product, we have

$$\frac{d^2R_0}{dVdU} = \frac{d^2R_0}{d\psi dU} = -\frac{\beta_2\mu}{a^3}\cos(\psi+h+g+l) \tag{1}$$

$$-\frac{1}{2}\frac{\beta_2\mu}{a^3}e\cos(\psi+h+g) \tag{2}$$

$$+2\frac{\beta_3\mu}{a^3}\gamma\cos(2\psi+2h+g+l). \tag{3}$$

In obtaining the product $\delta V \delta U$, it will be sufficient to take

$$\begin{aligned}
\delta V = & -3e'm\sin l' \\
& -\frac{9}{4}e'^2 m \sin 2l' \\
& +\left[\left(-\frac{3}{4}\gamma^2+\frac{75}{16}e^2\right)m+\frac{11}{8}m^2+\frac{59}{12}m^3\right]\sin(2h+2g+2l-2h'-2g'-2l') \\
& +\frac{15}{4}em\sin(2h+2g+l-2h'-2g'-2l') \\
& +\frac{45}{16}e^2m\sin(2h+2g-2h'-2g'-2l') \\
& +\frac{9}{4}\gamma^2m\sin(2h-2h'-2g'-2l'),
\end{aligned}$$

and

$$\begin{aligned}
\delta U = & \frac{3}{4}\gamma e'm\sin(g+l-l') \\
& -\frac{3}{4}\gamma e'm\sin(g+l+l') \\
& +\frac{11}{8}\gamma m^2\sin(2h+3g+3l-2h'-2g'-2l') \\
& +\frac{15}{4}\gamma em\sin(2h+3g+2l-2h'-2g'-2l') \\
& +\left[\frac{3}{4}\gamma m+\frac{25}{16}\gamma m^2\right]\sin(2h+g+l-2h'-2g'-2l') \\
& -\frac{3}{4}\gamma e'm\sin(2h+g+l-2h'-2g'-l') \\
& +\frac{3}{4}\gamma em\sin(2h+g+2l-2h'-2g'-2l') \\
& +3\gamma em\sin(2h+g-2h'-2g'-2l'),
\end{aligned}$$

and, preserving only the terms we need, the product is

$$\begin{aligned}
\delta V\delta U = & \left[\frac{9}{16}\gamma^3m^2+\frac{1845}{128}\gamma e^2m^2+\frac{9}{4}\gamma e'^2m^2+\frac{33}{64}\gamma m^3+\frac{989}{256}\gamma m^4\right]\cos(g+l) && (1) \\
& +\frac{45}{32}\gamma em^2\cos g && (2) \\
& +\frac{315}{128}\gamma e^2m^2\cos(g-l) && (3) \\
& -\frac{9}{8}\gamma e'm^2\cos(2h+g+l-2h'-2g'-3l') && (4) \\
& +\frac{9}{8}\gamma e'm^2\cos(2h+g+l-2h'-2g'-l') && (5) \\
& -\frac{9}{32}\gamma e'^2m^2\cos(2h+g+l-2h'-2g'). && (6)
\end{aligned}$$

On investigation, it is found that none of the six terms involving the squares and products of δr, δV, and δU contributes anything to the coefficient of the term whose argument is $\psi+2h+g-h'-g'$.

In arranging the periodic series for R, I adopt an order similar to that of DELAUNAY. Let $\zeta = \psi + h + g + l =$ the mean longitude of the moon counted from the moving mean equinox, and let D and F have the same significations as with DELAUNAY. The general form of the occurring argument is

$$k\zeta + k^{\mathrm{I}}\mathrm{D} + k^{\mathrm{II}}\mathrm{F} + k^{\mathrm{III}}l - k^{\mathrm{IV}}l',$$

k, k^{I}, k^{II}, k^{III} and k^{IV} being integers. We arrange our series in three divisions according as k is 0, 1 or 2. The following table exhibits the order; the columns to the right having the preference.

First Division, $k = 0$.

$k^{\mathrm{I}} = 0$,	$k^{\mathrm{II}} = 0$,	$k^{\mathrm{III}} = 0$,	$k^{\mathrm{IV}} = 0$,
$k^{\mathrm{I}} = 2$,	$k^{\mathrm{II}} = 2$,	$k^{\mathrm{III}} = 1$,	$k^{\mathrm{IV}} = 1$,
$k^{\mathrm{I}} = 1$,	$k^{\mathrm{II}} = -2$,	$k^{\mathrm{III}} = 2$,	$k^{\mathrm{IV}} = 2$,
		$k^{\mathrm{III}} = 3$,	$k^{\mathrm{IV}} = -1$,
		$k^{\mathrm{III}} = -1$,	$k^{\mathrm{IV}} = -2$.
		$k^{\mathrm{III}} = -2$,	

Second Division, $k = 1$.

$k^{\mathrm{I}} = 0$,	$k^{\mathrm{II}} = 1$,	$k^{\mathrm{III}} = 0$,	$k^{\mathrm{IV}} = 0$,
$k^{\mathrm{I}} = 2$,	$k^{\mathrm{II}} = 3$,	$k^{\mathrm{III}} = 1$,	$k^{\mathrm{IV}} = 1$,
$k^{\mathrm{I}} = -2$,	$k^{\mathrm{II}} = -1$,	$k^{\mathrm{III}} = 2$,	$k^{\mathrm{IV}} = 2$,
$k^{\mathrm{I}} = 1$,	$k^{\mathrm{II}} = -3$,	$k^{\mathrm{III}} = -1$,	$k^{\mathrm{IV}} = -1$,
$k^{\mathrm{I}} = -1$,		$k^{\mathrm{III}} = -2$,	$k^{\mathrm{IV}} = -2$.

Third Division, $k = 2$.

$k^{\mathrm{I}} = 0$,	$k^{\mathrm{II}} = 0$,	$k^{\mathrm{III}} = 0$,	$k^{\mathrm{IV}} = 0$,
$k^{\mathrm{I}} = 2$,	$k^{\mathrm{II}} = 2$,	$k^{\mathrm{III}} = 1$,	$k^{\mathrm{IV}} = 1$,
$k^{\mathrm{I}} = -2$,	$k^{\mathrm{II}} = -2$,	$k^{\mathrm{III}} = 2$,	$k^{\mathrm{IV}} = 2$,
$k^{\mathrm{I}} = -4$,		$k^{\mathrm{III}} = 3$,	$k^{\mathrm{IV}} = -1$,
$k^{\mathrm{I}} = 1$,		$k^{\mathrm{III}} = -1$,	$k^{\mathrm{IV}} = -2$.
$k^{\mathrm{I}} = -1$,		$k^{\mathrm{III}} = -2$,	
		$k^{\mathrm{III}} = -3$,	

In the designation of the source from which the portions of the following expression arise, the Roman numerals indicate which of the nine multiplications produces the terms in question. When no designation is given the terms belong to the elliptic value of R exhibited on pages 215–216.

$$\mathrm{R} = \beta_1 n'^2 \Big\{ \frac{1}{3} - 2\gamma^2 + \frac{1}{2}e^2 + 2\gamma^4 - 3\gamma^2 e^2 + \frac{5}{8}e^4 + \left(\frac{1}{6} - \gamma^2 + \frac{1}{12}e^2 + \frac{1}{4}e'^2\right)m^2 - \frac{179}{288}m^4$$

[I......1..

$$- \frac{97}{48}m^5 \cdot \quad \frac{7}{12}e^2m^3 - \frac{285}{64}e^2m^3 + \frac{225}{128}e^2m^2 + \frac{3765}{256}e^2m^3 + \frac{19}{36}m^4 + \frac{19}{6}m^5$$

........1] [I.......2.............4] [IV.....1...1]

$$+ \frac{15}{16}e^2m^3 - \frac{9}{32}\gamma^2m^3 - \frac{75}{64}\gamma^2m^3 + \frac{9}{4}\gamma^2m^3 \Big\}$$

[IV...2...2] [VI.....1..............1] [VIII...1...1]

(2) $$+\beta_1 n^2 \left\{ -\frac{3}{2} e' m^2 + \frac{21}{8} e^2 e' m - \frac{21}{8} e^2 e' m - \frac{3}{2} \gamma^2 e' m + \frac{3}{2} \gamma^2 e' m \right\} \cos l'$$
[I...1...2] [I...2...5] [I...2...6] [III...1...1] [III...1...3]

(3) $$+\beta_1 n^2 \left\{ -\frac{9}{4} e'^2 m^2 \right\} \cos 2l'$$
[I.. 1...3]

(4) $$+\beta_1 n^2 \left\{ e - 6\gamma^2 e + \frac{9}{8} e^3 - \frac{7}{12} e m^2 + \frac{1}{3} e m^2 \right\} \cos l$$
[I...1...4] [I...2...1]

(5) $$+\beta_1 n^2 \left\{ \frac{21}{8} e e' m \right\} \cos (l - l')$$
[I...1...5]

(6) $$+\beta_1 n^2 \left\{ -\frac{21}{8} e e' m \right\} \cos (l + l')$$
[I...1......6]

(7) $$+\beta_1 n^2 \left\{ \frac{3}{2} e^2 \right\} \cos 2l$$

(8) $$+\beta_1 n^2 \left\{ \frac{53}{24} e^3 \right\} \cos 3l$$

(9) $$+\beta_1 n^2 \left\{ 2\gamma^2 \right\} \cos (2g + 2l)$$

(10) $$+\beta_1 n^2 \left\{ 7\gamma^2 e \right\} \cos (2g + 3l)$$

(11) $$+\beta_1 n^2 \left\{ -\gamma^2 e - \frac{5}{2} \gamma^2 e \right\} \cos (2g + l)$$
[I..1..11]

(12) $$+\beta_1 n^2 \left\{ -\frac{5}{2} \gamma^2 e^2 + \frac{135}{16} \gamma^2 e^2 m - \frac{5}{2} \gamma^2 e^2 + \frac{135}{16} \gamma^2 e^2 m \right\} \cos 2g$$
[I........2.................11] [III....1..................7]

(13) $$+\beta_1 n^2 \left\{ \frac{15}{4} e^2 m + m^2 + \frac{19}{6} m^2 + \frac{15}{8} e^2 m + \frac{3}{2} \gamma^2 m \right\} \cos (2h + 2g + 2l - 2h' - 2g' - 2l')$$
[I.....1...................12] [I..2..18] [III..1..14]

(14) $$+\beta_1 n^2 \left\{ \frac{7}{2} e' m^2 \right\} \cos (2h + 2g + 2l - 2h' - 2g' - 3l')$$
[I..1..13]

(15) $$+\beta_1 n^2 \left\{ -\frac{1}{2} e' m^2 \right\} \cos (2h + 2g + 2l - 2h' - 2g' - l')$$
[I..1..15]

(16) $$+\beta_1 n^2 \left\{ \frac{33}{16} e m^2 + e m^2 \right\} \cos (2h + 2g + 3l - 2h' - 2g' - 2l')$$
[I..1..17] [I..2..12]

(17) $$+\beta_1 n^2 \left\{ \frac{15}{8} e m + \frac{187}{32} e m^2 + e m^2 \right\} \cos (2h + 2g + l - 2h' - 2g' - 2l')$$
[I....1............18] [I 2 12]

(18) $$+\beta_1 n^2 \left\{ \frac{35}{8} e e' m \right\} \cos (2h + 2g + l - 2h' - 2g' - 3l')$$
[I..1..19]

(19) $$+\beta_1 n^2 \left\{ -\frac{15}{8} e e' m \right\} \cos (2h + 2g + l - 2h' - 2g' - l')$$
[I....1....20]

(20) $$+ \beta_1 n^2 \left\{ -\frac{15}{4} e^2 m^3 + \frac{15}{8} e^2 m + \frac{187}{32} e^2 m^2 + \frac{5}{4} e^2 m^2 \right\} \cos (2h + 2g - 2h' - 2g' - 2l')$$
[I....1....22] [I......2..............18] [I..3..12]

(21) $$+ \beta_1 n^2 \left\{ \frac{35}{8} e^2 e' m \right\} \cos (2h + 2g - 2h' - 2g' - 3l')$$
[I...2...19]

(22) $$+ \beta_1 n^2 \left\{ -\frac{15}{8} e^2 e' m \right\} \cos (2h + 2g - 2h' - 2g' - l')$$
[I...2.....20]

(23) $$+ \beta_1 n^2 \left\{ -\frac{45}{32} e^2 e'^2 m \right\} \cos (2h + 2g - 2h' - 2g')$$
[I....2......21]

(24) $$+ \beta_1 n^2 \left\{ -3\gamma^2 m^2 + 3\gamma^2 m^2 - \frac{3}{2} \gamma^2 m - \frac{25}{8} \gamma^2 m^2 \right\} \cos (2h - 2h' - 2g' - 2l')$$
[I...1...23] [I..4..12] [III...1....... ....14]

(25) $$+ \beta_1 n^2 \left\{ -\frac{7}{2} \gamma^2 e' m \right\} \cos (2h - 2h' - 2g' - 3l')$$
[III...1...15]

(26) $$+ \beta_1 n^2 \left\{ \frac{3}{2} \gamma^2 e' m \right\} \cos (2h - 2h' - 2g' - l')$$
[III..1..17]

(27) $$+ \beta_1 n^2 \left\{ \frac{9}{8} \gamma^2 e'^2 m \right\} \cos (2h - 2h' - 2g')$$
[III...1....18]

(28) $$+ \beta_1 n^2 \left\{ -\frac{15}{16} m \frac{a}{a'} \right\} \cos (h + g + l - h' - g' - l')$$
[I...1....25]

(29) $$+ \beta_1 n^2 \left\{ \frac{5}{4} e' \frac{a}{a'} \right\} \cos (h + g + l - h' - g')$$
[I..1..26]

(30) $$+ \beta_1 n^2 \left\{ -\frac{15}{16} em \frac{a}{a'} \right\} \cos (h + g - h' - g' - l')$$
[I....2....25]

(31) $$+ \beta_1 n^2 \left\{ \frac{5}{4} ee' \frac{a}{a'} - \frac{45}{8} ee'm \frac{a}{a'} \right\} \cos (h + g - h' - g')$$
[I.....2................26]

(32) $$+ \beta_2 n^2 \left\{ \gamma - \frac{3}{2} \gamma^3 - \frac{5}{2} \gamma e^2 + \frac{1}{2} \gamma m^2 \right\} \cos (\psi + h + 2g + 2l)$$
[I...5...1]

(33) $$+ \beta_2 n^2 \left\{ \frac{3}{2} \gamma e' m + \frac{3}{8} \gamma e' m \right\} \cos (\psi + h + 2g + 2l - l')$$
[II..1..1] [III..2..1]

(34) $$+ \beta_2 n^2 \left\{ -\frac{3}{2} \gamma e' m - \frac{3}{8} \gamma e' m \right\} \cos (\psi + h + 2g + 2l + l')$$
[II...1....1] [III...2...3]

(35) $$+ \beta_2 n^2 \left\{ \frac{7}{2} \gamma e \right\} \cos (\psi + h + 2g + 3l)$$

(36) $$+ \beta_2 n^2 \left\{ \frac{17}{2} \gamma e^2 \right\} \cos (\psi + h + 2g + 4l)$$

(37) $$+ \beta_2 n^2 \left\{ -\frac{1}{2} \gamma e \right\} \cos (\psi + h + 2g + l)$$

(38)
$$+\beta_2 n^2 \Big\{ -\frac{5}{4}\gamma e^2 m^2 + \frac{7}{8}\gamma e^2 m^2 + \frac{15}{4}\gamma^3 e^2 + \frac{5}{8}\gamma^3 e^2 + \frac{7}{32}\gamma e^2 m^2 - \frac{5}{8}\gamma^3 e^2$$
[I...5......7] [I...6....4] [I...9...11] [II....1.................5] [II..3..12]
$$+\frac{15}{4}\gamma^3 e^2 - \frac{5}{8}\gamma e^2 - \frac{5}{2}\gamma^3 e^2 + \frac{47}{96}\gamma e^4 + \frac{135}{64}\gamma e^2 m + \frac{2025}{512}\gamma e^2 m^2 - \frac{25}{16}\gamma e^4$$
[II...5...11] [III....2..7] [III..3..8]
$$-\frac{5}{4}\gamma^3 e^2 + \frac{5}{16}\gamma e^4 + \frac{189}{128}\gamma e^2 m^2 - \frac{175}{128}\gamma e^2 m^2 - \frac{315}{256}\gamma e^2 m^2 - \frac{45}{128}\gamma e^2 m^2 \Big\}$$
[III.....4................................6] [VIII...2....2] [IX....1.....3] [IX....2.....2]
$$\times \cos(\psi + h + 2g)$$

(39)
$$+\beta_2 n^2 \Big\{ -\gamma + \frac{5}{2}\gamma^3 - \frac{3}{2}\gamma e^2 - \frac{7}{8}\gamma^5 + \frac{15}{4}\gamma^3 e^2 - \frac{15}{8}\gamma e^4 - \frac{15}{2}\gamma^3 e^2 + 3\gamma^3 m^2 + \frac{15}{4}\gamma^3 e^2$$
[I......5..............10] [I...6...11]
$$-\left(\frac{1}{2}\gamma + \frac{3}{4}\gamma e'^2\right)m^2 + \frac{179}{96}\gamma m^4 + \frac{5}{4}\gamma^3 m^2 - \frac{1}{4}\gamma e^2 m^2 + \frac{7}{8}\gamma e^2 m^2 + \frac{7}{8}\gamma e^2 m^2$$
[I.....7..1] [I. ..8....4] [I....9....4]
$$+\frac{25}{8}\gamma^3 e^2 - \frac{11}{8}\gamma^3 m^2 - \frac{5}{4}\gamma^3 e^2 + \frac{5}{8}\gamma e^2 m^2 + \frac{5}{4}\gamma^3 e^2 - \frac{5}{16}\gamma e^4 - \frac{189}{128}\gamma e^2 m^2 + \frac{25}{64}\gamma e^4$$
[II.......1...............8] [II..2..11] [III..3...5] [III.....4.............................6] [III..5..7]
$$-\frac{675}{128}\gamma e^2 m^2 - \frac{19}{12}\gamma m^4 + \frac{225}{64}\gamma e^2 m^2 + \frac{9}{4}\gamma e'^2 m^2 + \frac{121}{256}\gamma m^4 + \frac{9}{16}\gamma^3 m^2$$
[IV........3..................1] [V........1...................................1] [VI...2....1]
$$+\frac{135}{128}\gamma e^2 m^2 + \frac{9}{16}\gamma m^3 + \frac{123}{64}\gamma m^4 - \frac{9}{32}\gamma^3 m^2 - \frac{1845}{256}\gamma e^2 m^2 - \frac{9}{8}\gamma e'^2 m^2$$
[VIII.......2................................1] [IX.....1...... ....
$$-\frac{33}{128}\gamma m^3 - \frac{989}{512}\gamma m^4 - \frac{45}{128}\gamma e^2 m^2 \Big\} \cos(\psi + h)$$
..............................1] [IX....2....2]

(40)
$$+\beta_2 n^2 \Big\{ \frac{9}{4}\gamma e' m^2 - \frac{3}{2}\gamma e' m + \frac{3}{8}\gamma e' m + \frac{69}{64}\gamma e' m^2 \Big\} \cos(\psi + h - l')$$
[I...7...2] [II...3...1] [III....2................ ..3]

(41)
$$+\beta_2 n^2 \Big\{ -\frac{9}{8}\gamma e'^2 m + \frac{9}{32}\gamma e'^2 m \Big\} \cos(\psi + h - 2l')$$
[II....3.....2] [III....2....4]

(42)
$$+\beta_2 n^2 \Big\{ \frac{9}{4}\gamma e' m^2 + \frac{3}{2}\gamma e' m - \frac{3}{8}\gamma e' m - \frac{9}{64}\gamma e' m^2 \Big\} \cos(\psi + h + l')$$
[I...7...2] [II...3...1] [III.....2..................1]

(43)
$$+\beta_2 n^2 \Big\{ \frac{9}{8}\gamma e'^2 m - \frac{9}{32}\gamma e'^2 m \Big\} \cos(\psi + h + 2l')$$
[II..3...2] [III...2...2]

(44)
$$+\beta_2 n^2 \Big\{ -\frac{3}{2}\gamma e \Big\} \cos(\psi + h + l)$$

(45)
$$+\beta_2 n^2 \Big\{ -\frac{9}{4}\gamma e^2 + \frac{5}{8}\gamma e^2 \Big\} \cos(\psi + h + 2l)$$
[III..2..7]

(46)
$$+\beta_2 n^2 \Big\{ -\frac{3}{2}\gamma e \Big\} \cos(\psi + h - l)$$

(47)
$$+\beta_2 n^2 \Big\{ -\frac{9}{4}\gamma e^2 \Big\} \cos(\psi + h - 2l)$$

(48)
$$+\beta_2 n^2 \{ -\gamma^3 \} \cos(\psi + h - 2g - 2l)$$

(49) $$+\beta_2 n^2\left\{\frac{15}{4}\gamma^2e^2+\frac{5}{8}\gamma^2e^2-\frac{15}{4}\gamma^2e^2-\frac{5}{4}\gamma^2e^2+\frac{25}{4}\gamma^2e^2+\frac{25}{16}\gamma^2e^2\right\}\cos(\psi+h-2g)$$
[I..8..11] [II..3..12] [II..4..11] [III..2..10] [III..3..9] [III..6..7]

(50) $$+\beta_2 n^2\left\{\frac{3}{2}\gamma m^2+\frac{11}{16}\gamma m^2+\frac{11}{16}\gamma m^2\right\}\cos(\psi+3h+4g+4l-2h'-2g'-2l')$$
[I..5..12] [II..1..13] [III..2..11]

(51) $$+\beta_2 n^2\left\{\frac{45}{16}\gamma em+\frac{15}{8}\gamma em+\frac{15}{8}\gamma em\right\}\cos(\psi+3h+4g+3l-2h'-2g'-2l')$$
[I..5...18] [II..1...19] [III..2...12]

(52) $$+\beta_2 n^2\left\{-\frac{3}{2}\gamma m^2-\frac{11}{16}\gamma m^2+\frac{3}{8}\gamma m+\frac{25}{32}\gamma m^2\right\}\cos(\psi+3h+2g+2l-2h'-2g'-2l')$$
[I...7...12] [II..3...13] [III...2............14]

(53) $$+\beta_2 n^2\left\{\frac{7}{8}\gamma e'm\right\}\cos(\psi+3h+2g+2l-2h'-2g'-3l')$$
[III..2..15]

(54) $$+\beta_2 n^2\left\{-\frac{3}{8}\gamma e'm\right\}\cos(\psi+3h+2g+2l-2h'-2g'-l')$$
[III..2..17]

(55) $$+\beta_2 n^2\left\{\frac{3}{8}\gamma em+\frac{15}{16}\gamma em\right\}\cos(\psi+3h+2g+3l-2h'-2g'-2l')$$
[III..2..19] [III..3..14]

(56) $$+\beta_2 n^2\left\{-\frac{45}{16}\gamma em-\frac{15}{8}\gamma em+\frac{3}{2}\gamma em+\frac{3}{16}\gamma em\right\}\cos(\psi+3h+2g+l-2h'-2g'-2l')$$
[I....7....18] [II...3...19] [III..2..20] [III..4..14]

(57) $$+\beta_2 n^2\left\{-\frac{45}{16}\gamma e^2m-\frac{45}{32}\gamma e^2m-\frac{45}{16}\gamma e^2m+\frac{147}{64}\gamma e^2m+\frac{3}{4}\gamma e^2m+\frac{15}{64}\gamma e^2m\right\}$$
[I....9.....18] [II...3....23] [II...5....19] [III...2....21] [III..4..20] [III..5...14]
$$\times\cos(\psi+3h+2g-2h'-2g'-2l')$$

(58) $$+\beta_2 n^2\left\{-\frac{9}{8}\gamma^3m+\frac{15}{16}\gamma^3m+\frac{15}{16}\gamma^3m\right\}\cos(\psi+3h-2h'-2g'-2l')$$
[II..3....24] [III..2..22] [III..6...14]

(59) $$+\beta_2 n^2\left\{\frac{3}{2}\gamma m^2+\frac{45}{8}\gamma e^2m+\frac{19}{4}\gamma m^3-\frac{45}{16}\gamma e^2m+\left(\frac{3}{8}\gamma^3-\frac{75}{32}\gamma e^2\right)m-\frac{11}{16}\gamma m^2-\frac{59}{24}\gamma m^3\right.$$
[I....5..............................12] [I...6....18] [II............1..13]
$$+\frac{15}{16}\gamma e^2m+\frac{9}{8}\gamma^3m-\left(\frac{3}{8}\gamma-\frac{15}{16}\gamma^3+\frac{33}{32}\gamma e^2-\frac{15}{16}\gamma e'^2\right)m-\frac{25}{32}\gamma m^2-\frac{2957}{1536}\gamma m^3$$
[II...2....19] [II..3..24] [III.........2..14]
$$\left.-\frac{15}{16}\gamma e^2m-\frac{3}{4}\gamma e^2m-\frac{3}{16}\gamma m^3\right\}\cos(\psi-h+2h'+2g'+2l')$$
[III..3..19] [III..4..20] [VIII..2..3]

(60) $$+\beta_2 n^2\left\{-\frac{3}{4}\gamma e'm^2+\frac{11}{32}\gamma e'm^2+\frac{3}{8}\gamma e'm+\frac{115}{64}\gamma e'm^2-\frac{9}{16}\gamma e'm^2\right\}$$
[I...5...15] [II...1.....16] [III....2...................17] [IX....1....5]
$$\times\cos(\psi-h+2h'+2g'+l')$$

(61) $$+\beta_2 n^2\left\{\frac{9}{32}\gamma e'^2m+\frac{57}{256}\gamma e'^2m^2+\frac{9}{64}\gamma e'^2m^2\right\}\cos(\psi-h+2h'+2g')$$
[III....2....................18] [IX...1....6]

(62) $$+\beta_2 n^2\left\{\frac{21}{4}\gamma e'm^2-\frac{77}{32}\gamma e'm^2-\frac{7}{8}\gamma e'm-\frac{255}{64}\gamma e'm^2+\frac{9}{16}\gamma e'm^2\right\}$$
[I...5....13] [II...1...14] [III......2................15] [IX....1....4]
$$\times\cos(\psi-h+2h'+2g'+3l')$$

(63) $$+\beta_2 n^2\left\{-\frac{51}{32}\gamma e'^2m\right\}\cos(\psi-h+2h'+2g'+4l')$$
[III....2....16]

$$(64)\qquad +\beta_2 n^2 \left\{ \frac{45}{16}\gamma em - \frac{15}{8}\gamma em - \frac{3}{2}\gamma em - \frac{15}{16}\gamma em \right\} \cos(\psi - h + l + 2h' + 2g' + 2l')$$
[I..5...18] [II..1...19] [III..2..20] [III..3..14]

$$(65)\qquad +\beta_2 n^2 \left\{ -\frac{3}{8}\gamma em - \frac{3}{16}\gamma em \right\} \cos(\psi - h - l + 2h' + 2g' + 2l')$$
[III..2...19] [III..4...14]

$$(66)\qquad +\beta_2 n^2 \left\{ -\frac{3}{2}\gamma m^2 + \frac{11}{16}\gamma m^2 - \frac{11}{16}\gamma m^2 \right\} \cos(\psi - h - 2g - 2l + 2h' + 2g' + 2l')$$
[I...7...12] [II..3....13] [III..2...11]

$$(67)\qquad +\beta_2 n^2 \left\{ -\frac{45}{16}\gamma em + \frac{15}{8}\gamma em - \frac{15}{8}\gamma em \right\} \cos(\psi - h - 2g - l + 2h' + 2g' + 2l')$$
[I....7.....18] [II...3...19] [III..2..12]

$$(68)\qquad +\beta_2 n^2 \left\{ -\frac{45}{16}\gamma e^2 m + \frac{45}{32}\gamma e^2 m + \frac{45}{16}\gamma e^2 m + \frac{15}{64}\gamma e^2 m - \frac{75}{16}\gamma e^2 m \right\}$$
[I....8.....18] [II...3....23] [II...4....19] [III..2....13] [III..3...12]
$$\times \cos(\psi - h - 2g + 2h' + 2g' + 2l')$$

$$(69)\qquad +\beta_2 n^2 \left\{ -\frac{15}{8}\gamma ee' \frac{a}{a'} + \frac{135}{16}\gamma ee'm \frac{a}{a'} - \frac{25}{16}\gamma ee' \frac{a}{a'} + \frac{495}{32}\gamma ee'm \frac{a}{a'} - \frac{15}{8}\gamma ee' \frac{a}{a'} \right.$$
[I....... ...9........... ....26] [II........3......................31] [II.......5...
$$\left. + \frac{135}{16}\gamma ee'm \frac{a}{a'} + \frac{25}{16}\gamma ee' \frac{a}{a'} - \frac{955}{32}\gamma ee'm \frac{a}{a'} + \frac{5}{8}\gamma ee' \frac{a}{a'} - \frac{45}{16}\gamma ee'm \frac{a}{a'} \right\}$$
..................28] [III....... 2......................27] [III.......4.................25]
$$\times \cos(\psi + 2h + g - h' - g')$$

$$(70)\qquad +\beta_2 n^2 \left\{ \frac{15}{4}\gamma ee' \frac{a}{a'} - \frac{15}{8}\gamma ee' \frac{a}{a'} - \frac{25}{16}\gamma ee' \frac{a}{a'} + \frac{5}{8}\gamma ee' \frac{a}{a'} - \frac{55}{48}\gamma ee' \frac{a}{a'} - \frac{5}{8}\gamma ee' \frac{a}{a'} \right\}$$
[I...5....28] [I....6....26] [II...1....30] [II..2...28] [III...2...26] [III..4...25]
$$\times \cos(\psi + g + h' + g')$$

$$(71)\qquad +\beta_2 n^2 \left\{ -\frac{15}{8}\gamma ee' \frac{a}{a'} + \frac{25}{16}\gamma ee' \frac{a}{a'} + \frac{15}{8}\gamma ee' \frac{a}{a'} + \frac{5}{16}\gamma ee' \frac{a}{a'} - \frac{25}{8}\gamma ee' \frac{a}{a'} \right\}$$
[I....8....26] [II...3....31] [II...4....28] [III..2.....24] [III...3....23]
$$\times \cos(\psi - g + h' + g')$$

$$(72)\qquad +\beta_3 n^2 \left\{ \frac{1}{2} - \gamma^2 - \frac{5}{4}e^2 + \frac{1}{4}m^2 \right\} \cos(2\psi + 2h + 2g + 2l)$$
[I..10..1]

$$(73)\qquad +\beta_3 n^2 \left\{ -\frac{9}{8}e'm^2 + \frac{3}{2}e'm \right\} \cos(2\psi + 2h + 2g + 2l - l')$$
[I...10...2] [II..6...1]

$$(74)\qquad +\beta_3 n^2 \left\{ \frac{9}{8}e'^2 m \right\} \cos(2\psi + 2h + 2g + 2l - 2l')$$
[II..6..2]

$$(75)\qquad +\beta_3 n^2 \left\{ -\frac{9}{8}e'm^2 - \frac{3}{2}e'm \right\} \cos(2\psi + 2h + 2g + 2l + l')$$
[I...10...2] [II..6..1]

$$(76)\qquad +\beta_3 n^2 \left\{ -\frac{9}{8}e'^2 m \right\} \cos(2\psi + 2h + 2g + 2l + 2l')$$
[II...6...2]

$$(77)\qquad +\beta_3 n^2 \left\{ \frac{7}{4}e - \frac{7}{2}\gamma^2 e - \frac{123}{32}e^3 - \frac{7}{16}em^2 + \frac{3}{4}em^2 \right\} \cos(2\psi + 2h + 2g + 3l)$$
[I..10...4] [I..11..1]

$$(78)\qquad +\beta_3 n^2 \left\{ \frac{63}{32}ee'm + \frac{21}{8}ee'm + \frac{21}{4}ee'm \right\} \cos(2\psi + 2h + 2g + 3l - l')$$
[I..10...5] [II...6...3] [II...7...1]

(79) $$+\beta_3 n^2 \left\{ -\frac{63}{32} ee'm - \frac{21}{8} ee'm - \frac{21}{4} ee'm \right\} \cos(2\psi + 2h + 2g + 3l + l')$$
[I....10.....6] [II..6...4] [II...7...1]

(80) $$+\beta_3 n^2 \left\{ \frac{17}{4} e^3 \right\} \cos(2\psi + 2h + 2g + 4l)$$

(81) $$+\beta_3 n^2 \left\{ \frac{845}{96} e^3 \right\} \cos(2\psi + 2h + 2g + 5l)$$

(82) $$+\beta_3 n^2 \left\{ -\frac{1}{4} e + \frac{1}{2}\gamma^2 e + \frac{1}{32} e^3 - \frac{7}{16} em^2 - \frac{1}{4} em^2 \right\} \cos(2\psi + 2h + 2g + l)$$
[I..10...4] [I..12..1]

(83) $$+\beta_3 n^2 \left\{ -\frac{63}{32} ee'm + \frac{21}{8} ee'm - \frac{3}{4} ee'm \right\} \cos(2\psi + 2h + 2g + l - l')$$
[I....10....6] [II...6...4] [II..8...1]

(84) $$+\beta_3 n^2 \left\{ \frac{63}{32} ee'm - \frac{21}{8} ee'm + \frac{3}{4} ee'm \right\} \cos(2\psi + 2h + 2g + l + l')$$
[I..10...5] [II...6...3] [II..8...1]

(85) $$+\beta_3 n^2 \left\{ -\frac{5}{8} e^2m^2 - \frac{2205}{256} e^2m^3 + \frac{7}{16} e^2m^2 - \frac{855}{256} e^2m^3 + \frac{5}{8}\gamma^2 e^2 - \frac{135}{64}\gamma^2 e^2 m + \frac{7}{32} e^2m^2 \right.$$
[I........10................7] [I........12............ 4] [II....... ....6.................... .

$$+\frac{2595}{512} e^2m^3 + \frac{5}{8}\gamma^2 e^2 - \frac{135}{64}\gamma^2 e^2 m + \frac{1485}{512} e^2m^3 - \frac{135}{64} e^2m^3 - \frac{1515}{256} e^2m^3$$
..............5] [III........8..............7] [IV....4.....3] [IV..5...2] [V....2......3]

$$\left. + \frac{165}{128} e^2m^3 - \frac{225}{256} e^2m^3 + \frac{585}{128} e^2m^3 \right\} \cos(2\psi + 2h + 2g)$$
[V...3....2] [VII..2....1] [VII..1....2]

(86) $$+\beta_3 n^2 \left\{ -\frac{63}{16} e^2e'm + \frac{63}{32} e^2e'm + \frac{105}{32} e^2e'm - \frac{21}{16} e^2e'm \right\} \cos(2\psi + 2h + 2g - l)$$
[I....10......9] [I...12...6] [II....6.....7] [II...8....4]

(87) $$+\beta_3 n^2 \left\{ \frac{63}{16} e^2e'm - \frac{63}{32} e^2e'm - \frac{105}{32} e^2e'm + \frac{21}{16} e^2e'm \right\} \cos(2\psi + 2h + 2g + l'$$
[I...10...8] [I...12....5] [II....6....6] [II....8....3]

(88) $$+\beta_3 n^2 \left\{ \frac{1}{96} e^3 \right\} \cos(2\psi + 2h + 2g - l)$$

(89) $$+\beta_3 n^2 \left\{ -\frac{15}{8}\gamma^2 e - \frac{5}{2}\gamma^2 e \right\} \cos(2\psi + 2h + 4g + 3l)$$
[I...10 .11] [II..6..11]

(90) $$+\beta_3 n^2 \left\{ \gamma^2 - \gamma^4 + \frac{3}{2}\gamma^2 e^2 - \frac{15}{4}\gamma^2 e^2 + \frac{405}{32}\gamma^2 e^2 m + \frac{3}{2}\gamma^2 m^2 - \frac{9}{4}\gamma^2 m^3 + \frac{15}{8}\gamma^2 e^2 \right.$$
[I... ....10.................. ....10] [I...12...

$$-\frac{405}{64}\gamma^2 e^2 m + \frac{1}{2}\gamma^2 m^2 + \frac{25}{8}\gamma^2 e^2 - \frac{675}{64}\gamma^2 e^2 m - \frac{11}{8}\gamma^2 m^2 + \frac{231}{128}\gamma^2 m^3$$
.... [I..13...1] [II..............6....... ... .8]

$$\left. -\frac{5}{4}\gamma^2 e^2 + \frac{135}{32}\lambda^2 e^2 m - \frac{99}{64}\gamma^2 m^3 - \frac{33}{128}\gamma^2 m^3 + \frac{27}{16}\gamma^2 m^3 - \frac{9}{16}\gamma^2 m^3 + \frac{33}{64}\gamma^2 m^3 \right\}$$
[II.........8.... .. ..11] [V...2....4] [VI...3....2] [VII..1...3] [VIII..3..1] [IX..3...1]

$$\times \cos(2\psi + 2h)$$

(91) $$+\beta_3 n^2 \left\{ -\frac{3}{8}\gamma^2 e'm + 3\gamma^2 e'm - \frac{3}{8}\gamma^2 e'm \right\} \cos(2\psi + 2h - l')$$
[II....6....10] [II..9...1] [III...8...3]

(92) $$+\beta_3 n^2 \left\{ \frac{3}{8}\gamma^2 e'm - 3\gamma^2 e'm + \frac{3}{8}\gamma^2 e'm \right\} \cos(2\psi + 2h + l')$$
[II..6...9] [II..9...1] [III..8...1]

(93) $$+\beta_3 n^2 \left\{ \frac{3}{2}\gamma^2 e - \frac{15}{8}\gamma^2 e + \frac{5}{2}\gamma^2 e \right\} \cos(2\psi + 2h + l)$$
[I..10..11] [II..6..11]

(94) $$+\beta_3 n^2\left\{\frac{3}{2}\gamma^2 e\right\}\cos(2\psi+2h-l)$$

(95) $$+\beta_3 n^2\left\{\frac{45}{16}e^2m+\frac{3}{4}m^2+\frac{19}{8}m^3+\frac{135}{32}e^2m-\left(\frac{3}{8}\gamma^2-\frac{75}{32}e^2\right)m+\frac{11}{16}m^2+\frac{59}{24}m^3\right.$$
[I......10....................12] [I..11...18] [II........... 6........... ....13]

$$\left.+\frac{105}{16}e^2m+\frac{3}{8}\gamma^2m\right\}\cos(2\psi+4h+4g+4l-2h'-2g'-2l')$$
[II....7....19] [III..7..14]

(96) $$+\beta_3 n^2\left\{\frac{21}{8}e'm^2+\frac{77}{32}e'm^2\right\}\cos(2\psi+4h+4g+4l-2h'-2g'-3l')$$
[I...10..13] [II..6..14]

(97) $$+\beta_3 n^2\left\{-\frac{3}{8}e'm^2-\frac{11}{32}e'm^2\right\}\cos(2\psi+4h+4g+4l-2h'-2g'-l')$$
[I...10...15] [II...6...16]

(98) $$+\beta_3 n^2\left\{\frac{99}{64}em^2+\frac{9}{4}em^2+\frac{17}{16}em^2+\frac{77}{32}em^2\right\}\cos(2\psi+4h+4g+5l-2h'-2g'-2l')$$
[I..10..17] [I..11..12] [II..6..18] [II..7..13]

(99) $$+\beta_3 n^2\left\{\frac{45}{32}em+\frac{561}{128}em^2-\frac{3}{4}em^2+\frac{15}{8}em+\frac{263}{32}em^2-\frac{11}{32}em^2\right\}$$
[I........10.........18] [I..12..12] [II......6 ....19] [II..8..13]

$$\times\cos(2\psi+4h+4g+3l-2h'-2g'-2l')$$

(100) $$+\beta_3 n^2\left\{\frac{105}{32}ee'm+\frac{35}{8}ee'm\right\}\cos(2\psi+4h+4g+3l-2h'-2g'-3l')$$
[I...10...19] [II..6...20]

(101) $$+\beta_3 n^2\left\{-\frac{45}{32}ee'm-\frac{15}{8}ee'm\right\}\cos(2\psi+4h+4g+3l-2h'-2g'-l')$$
[I...10....20] [II..6...21]

(102) $$+\beta_3 n^2\left\{-\frac{45}{32}e^2m+\frac{45}{32}e^2m-\frac{15}{16}e^2m\right\}\cos(2\psi+4h+4g+2l-2h'-2g'-2l')$$
[I...12....18] [II..6...23] [II..8...19]

(103) $$+\beta_3 n^2\left\{\frac{9}{8}\gamma^2m-\frac{3}{8}\gamma^2m\right\}\cos(2\psi+4h+2g+2l-2h'-2g'-2l')$$
[II..6..24] [III..8..14]

(104) $$+\beta_3 n^2\left\{\frac{45}{16}e^2m+\left(\frac{3}{4}-3\gamma^2+\frac{399}{64}e^2-\frac{15}{8}e'^2\right)m^2+\frac{19}{8}m^3+\frac{131}{24}m^4+\frac{297}{64}e^2m^2\right.$$
[I........10..12] [I..11....17]

$$-\frac{45}{32}e^2m-\frac{561}{128}e^2m^2+\left(\frac{3}{8}\gamma^2-\frac{75}{32}e^2\right)m-\left(\frac{11}{16}-\frac{91}{32}\gamma^2+\frac{881}{128}e^2-\frac{55}{32}e'^2\right)m^2$$
[I...........12..... ......18] [II............6........ .. ..

$$-\frac{59}{24}m^3-\frac{893}{144}m^4-\frac{119}{32}e^2m^2+\frac{15}{16}e^2m+\frac{263}{64}e^2m^2-\frac{11}{16}\gamma^2m^2+\frac{3}{8}\gamma^2m$$
......................13] [II...7....18] [II.........8............19] [III..7...11] [III..8....

$$\left.+\frac{25}{32}\gamma^2m^2+\frac{1}{4}m^4-\frac{11}{32}m^4\right\}\cos(2\psi+2h'+2g'+2l')$$
........14] [IV..4..4] [VII..1..4]

(105) $$+\beta_3 n^2\left\{-\frac{45}{16}e^2e'm-\frac{3}{8}e'm^2-\frac{91}{32}e'm^3+\frac{45}{32}e^2e'm-\left(\frac{3}{8}\gamma^2e'-\frac{75}{32}e^2e'\right)m+\frac{11}{32}e'm^2\right.$$
[I...........10........................15] [I..12....20] [II........6....... ....

$$\left.+\frac{257}{96}e'm^3-\frac{15}{16}e^2e'm-\frac{3}{8}\gamma^2e'm-\frac{33}{16}e'm^3+\frac{9}{4}e'm^3\right\}\cos(2\psi+2h'+2g'+l')$$
...........16] [II, 8....21] [III..8...17] [V..2....6] [VII..1..6]

(106) $$+\beta_3 n^2 \Big\{ -\frac{135}{64} e^2 e'^2 m - \frac{9}{16} e'^2 m^3 + \frac{135}{128} e^2 e'^2 m - \left(\frac{9}{32}\gamma^2 e'^2 - \frac{225}{128} e^2 e'^2\right) m + \frac{33}{64} e'^2 m^3$$

[I.........10................16] [I....12.....21] [II..........6............ ....17]

$$-\frac{45}{64} e^2 e'^2 m - \frac{9}{32}\gamma^2 e'^2 m - \frac{33}{64} e'^2 m^3 + \frac{9}{16} e'^2 m^3 \Big\} \cos(2\psi + 2h' + 2g')$$

[II....8.....22] [III...8....18] [V...2....7] [VII..1...7]

(107) $$+\beta_3 n^2 \Big\{ \frac{105}{16} e^2 e' m + \frac{21}{8} e' m^2 + \frac{471}{32} e' m^3 - \frac{105}{32} e^2 e' m + \left(\frac{7}{8}\gamma^2 e' - \frac{175}{32} e^2 e'\right) m$$

[I........10............................13] [I... 12....19] [II........6....................

$$-\frac{77}{32} e' m^2 - \frac{479}{32} e' m^3 + \frac{35}{16} e^2 e' m + \frac{7}{8}\gamma^2 e' m + \frac{33}{16} e' m^3 - \frac{9}{4} e' m^3 \Big\}$$

.........................14] [II..8...20] [III..8..15] [V...2...5] [VII..1..5]

$$\times \cos(2\psi + 2h' + 2g' + 3l')$$

(108) $$+\beta_3 n^2 \Big\{ \frac{51}{8} e'^2 m^2 - \frac{187}{32} e'^2 m^2 \Big\} \cos(2\psi + 2h' + 2g' + 4l')$$

[I..10..14] [II...6....15]

(109) $$+\beta_3 n^2 \Big\{ \frac{45}{32} em + \frac{561}{128} em^2 + \frac{9}{4} em^2 - \frac{15}{8} em - \frac{263}{32} em^2 - \frac{77}{32} em^2 \Big\}$$

[I........10.........18] [I..11..12] [II........6..........19] [II..7...13]

$$\times \cos(2\psi + l + 2h' + 2g' + 2l')$$

(110) $$+\beta_3 n^2 \Big\{ -\frac{45}{32} ee'm + \frac{15}{8} ee'm \Big\} \cos(2\psi + l + 2h' + 2g' + l')$$

[I....10....20] [II..6...21]

(111) $$+\beta_3 n^2 \Big\{ \frac{105}{32} ee'm - \frac{35}{8} ee'm \Big\} \cos(2\psi + l + 2h' + 2g' + 3l')$$

[I..10...19] [II..6...20]

(112) $$+\beta_3 n^2 \Big\{ \frac{135}{32} e^2 m - \frac{45}{32} e^2 m - \frac{105}{16} e^2 m \Big\} \cos(2\psi + 2l + 2h' + 2g' + 2l')$$

[I..11..18] [II..6..23] [II..7..19]

(113) $$+\beta_3 n^2 \Big\{ \frac{99}{64} em^2 - \frac{3}{4} em^2 - \frac{17}{16} em^2 + \frac{11}{32} em^2 \Big\} \cos(2\psi - l + 2h' + 2g' + 2l')$$

[I..10..17] [I..12..12] [II..6..18] [II..8..13]

(114) $$+\beta_3 n^2 \Big\{ -\frac{9}{8}\gamma^2 m - \frac{3}{8}\gamma^2 m \Big\} \cos(2\psi + 2g + 2l + 2h' + 2g' + 2l')$$

[II...6...24] [III..7..14]

(115) $$+\beta_3 n^2 \Big\{ \frac{675}{256} e^2 m^2 - \frac{1125}{512} e^2 m^2 + \frac{675}{512} e^2 m^2 - \frac{675}{128} e^2 m^2 \Big\} \cos(2\psi - 2h - 2g + 4h' + 4g' + 4l')$$

[I..10..14] [II....6....25] [IV...4....6] [VII..1...8]

(116) $$+\beta_3 n^2 \Big\{ \frac{9}{128}\gamma^2 m^2 + \frac{9}{128}\gamma^2 m^2 \Big\} \cos(2\psi - 2h + 4h' + 4g' + 4l')$$

[II...6...26] [VI...3....3]

(117) $$+\beta_3 n^2 \Big\{ -\frac{45}{64} m \frac{a}{a'} - \frac{15}{16} m \frac{a}{a'} \Big\} \cos(2\psi + 3h + 3g + 3l - h' - g' - l')$$

[I....10....25] [II..6...27]

(118) $$+\beta_3 n^2 \Big\{ \frac{15}{16} e' \frac{a}{a'} + \frac{5}{4} e' \frac{a}{a'} \Big\} \cos(2\psi + 3h + 3g + 3l - h' - g')$$

[I..10..26] [II..6..28]

(119) $$+\beta_3 n^2 \Big\{ -\frac{45}{64} m \frac{a}{a'} + \frac{15}{16} m \frac{a}{a'} \Big\} \cos(2\psi + h + g + l + h' + g' + l')$$

[I...10. . 25] [II..6...27]

(120) $$+\beta_3 n^2 \Big\{ \frac{15}{16} e' \frac{a}{a'} - \frac{5}{4} e' \frac{a}{a'} \Big\} \cos(2\psi + h + g + l + h' + g')$$

[I..10..26] [II..6..28]

$$(121) \qquad + \beta_3 n^2 \left\{ -\frac{45}{32} em \frac{a}{a'} + \frac{45}{64} em \frac{a}{a'} + \frac{75}{64} em \frac{a}{a'} - \frac{15}{32} em \frac{a}{a'} \right\} \cos(2\psi + h + g + h' + g' + l')$$

[I....10....27] [I....12....25] [II....6....29] [II..8....27]

$$(122) \qquad + \beta_3 n^2 \left\{ \frac{15}{8} ee' \frac{a}{a'} - \frac{135}{8} ee'm \frac{a}{a'} - \frac{15}{16} ee' \frac{a}{a'} + \frac{135}{32} ee'm \frac{a}{a'} - \frac{25}{16} ee' \frac{a}{a'} + \frac{225}{32} ee'm \frac{a}{a'} \right.$$

[I........10................28] [I........12..................26] [II.... ...6... .... .30]

$$\left. + \frac{5}{8} ee' \frac{a}{a'} - \frac{45}{16} ee'm \frac{a}{a'} \right\} \cos(2\psi + h + g + h' + g').$$

[II........8...... ....28]

Before giving the reduced value of the preceding expression, we note that the signification of the symbols a, e and γ it contains are those of DELAUNAY after the transformation of Tom. II, p. 800. If these variables should be retained, the final expressions for $\frac{da}{dL}$, $\frac{da}{dG}$, $\frac{da}{dH}$, &c., given by DELAUNAY, would need modification. On trial, this is found to complicate these expressions so much, that it appears a saving of labor would be effected by reverting to DELAUNAY's variables, such as they were before the transformation just mentioned. Consequently, after summing the various parts of the coefficients of the preceding expression, we make the following transformation, the reverse of that given by DELAUNAY (Tom. II, p. 800). We replace

$$a \text{ by } a \left\{ 1 + \left(\frac{2}{3} - 3\gamma^2 + \frac{3}{4} e^2 + e'^2 \right) \frac{n'^2}{n^2} + \left(\frac{9}{4} \gamma^2 + \frac{225}{16} e^2 \right) \frac{n'^3}{n^3} - \frac{1193}{288} \frac{n'^4}{n^4} - \frac{787}{48} \frac{n'^5}{n^5} \right\},$$

$$e \text{ by } e \left\{ 1 - \frac{81}{128} \frac{n'^2}{n^2} + \frac{2595}{256} \frac{n'^3}{n^3} \right\},$$

$$\gamma \text{ by } \gamma \left\{ 1 - \frac{5}{8} \gamma^2 e^2 - \frac{15}{128} e^4 - \left(\frac{57}{128} - \frac{293}{128} \gamma^2 + \frac{991}{256} e^2 + \frac{103}{128} e'^2 \right) \frac{n'^2}{n^2} + \frac{129}{256} \frac{n'^3}{n^3} - \frac{229}{32768} \frac{n'^4}{n^4} \right\}.$$

In order to be reminded that this change has been made, we shall discard m, writing everywhere $\frac{n'}{n}$ in its place. It will be noted that this change affects only those coefficients which have three or more different orders of quantities in their terms; that is, the coefficients of the terms numbered (1), (4), (32), (38), (39), (59), (72), (77), (82), (90), (104).

The following, then, is the reduced expression for R:

$$R = \beta_1 n^2 \left\{ \frac{1}{3} - 2\gamma^2 + \frac{1}{2} e^2 + 2\gamma^4 - 3\gamma^2 e^2 + \frac{5}{8} e^4 + \left(-\frac{1}{2} + \frac{15}{2} \gamma^2 - \frac{9}{8} e^2 - \frac{3}{4} e'^2 \right) \frac{n'^2}{n^2} \right.$$

$$+ \left(-\frac{51}{16} \gamma^2 + \frac{465}{64} e^2 \right) \frac{n'^3}{n^3} + \frac{79}{16} \frac{n'^4}{n^4} + \frac{421}{24} \frac{n'^5}{n^5}$$

$$(2) \qquad - \frac{3}{2} e' \frac{n'^2}{n^2} \cos l'$$

$$(3) \qquad - \frac{9}{4} e'^2 \frac{n'^2}{n^2} \cos 2l'$$

$$(4) \qquad + \left[e - 6\gamma^2 e + \frac{9}{8} e^3 - \frac{369}{128} e \frac{n'^2}{n^2} \right] \cos l$$

$$(5) \qquad + \frac{21}{8} ee' \frac{n'}{n} \cos(l - l')$$

$$(6) \qquad - \frac{21}{8} ee' \frac{n'}{n} \cos(l + l')$$

(7) $+\frac{3}{2}e^2\cos 2l$

(8) $+\frac{53}{24}e^3\cos 3l$

(9) $+2\gamma^2\cos(2g+2l)$

(10) $+7\gamma^2 e\cos(2g+3l)$

(11) $-\frac{7}{2}\gamma^2 e\cos(2g+l)$

(12) $+\left[-5\gamma^2 e^2+\frac{135}{8}\gamma^2 e^2\frac{n'}{n}\right]\cos 2g$

(13) $+\left[\left(\frac{3}{2}\gamma^2+\frac{45}{8}e^2\right)\frac{n'}{n}+\frac{n'^2}{n^2}+\frac{19}{6}\frac{n'^3}{n^3}\right]\cos(2h+2g+2l-2h'-2g'-2l')$

(14) $+\frac{7}{2}e'\frac{n'^2}{n^2}\cos(2h+2g+2l-2h'-2g'-3l')$

(15) $-\frac{1}{2}e'\frac{n'^2}{n^2}\cos(2h+2g+2l-2h'-2g'-l')$

(16) $+\frac{49}{16}e\frac{n'^2}{n^2}\cos(2h+2g+3l-2h'-2g'-2l')$

(17) $+\left[\frac{15}{8}e\frac{n'}{n}+\frac{219}{32}e\frac{n'^2}{n^2}\right]\cos(2h+2g+l-2h'-2g'-2l')$

(18) $+\frac{35}{8}ee'\frac{n'}{n}\cos(2h+2g+l-2h'-2g'-3l')$

(19) $-\frac{15}{8}ee'\frac{n'}{n}\cos(2h+2g+l-2h'-2g'-l')$

(20) $+\left[\frac{15}{8}e^2\frac{n'}{n}+\frac{107}{32}e^2\frac{n'^2}{n^2}\right]\cos(2h+2g-2h'-2g'-2l')$

(21) $+\frac{35}{8}e^2e'\frac{n'}{n}\cos(2h+2g-2h'-2g'-3l')$

(22) $-\frac{15}{8}e^2e'\frac{n'}{n}\cos(2h+2g-2h'-2g'-l')$

(23) $-\frac{45}{32}e^2e'^2\frac{n'}{n}\cos(2h+2g-2h'-2g')$

(24) $-\left[\frac{3}{2}\gamma^2\frac{n'}{n}+\frac{25}{8}\gamma^2\frac{n'^2}{n^2}\right]\cos(2h-2h'-2g'-2l')$

(25) $-\frac{7}{2}\gamma^2 e'\frac{n'}{n}\cos(2h-2h'-2g'-3l')$

(26) $+\frac{3}{2}\gamma^2 e'\frac{n'}{n}\cos(2h-2h'-2g'-l')$

(27) $+\frac{9}{8}\gamma^2 e'^2\frac{n'}{n}\cos(2h-2h'-2g')$

(28) $-\frac{15}{16}\frac{n'}{n}\frac{a}{a'}\cos(h+g+l-h'-g'-l')$

(29) $+\frac{5}{4}e'\frac{a}{a'}\cos(h+g+l-h'-g')$

(30) $-\frac{15}{16}e\frac{n'}{n}\frac{a}{a'}\cos(h+g-h'-g'-l')$

(31) $$+\left[\frac{5}{4}ee'\frac{a}{a'}-\frac{45}{8}ee'\frac{n'}{n}\frac{a}{a'}\right]\cos(h+g-h'-g')\Big\}$$

(32) $$+\beta_2 n^2\Big\{\left[\gamma-\frac{3}{2}\gamma^3-\frac{5}{2}\gamma e^2-\frac{249}{128}\gamma\frac{n'^2}{n^2}\right]\cos(\psi+h+2g+2l)$$

(33) $$+\frac{15}{8}\gamma e'\frac{n'}{n}\cos(\psi+h+2g+2l-l')$$

(34) $$-\frac{15}{8}\gamma e'\frac{n'}{n}\cos(\psi+h+2g+2l+l')$$

(35) $$+\frac{7}{2}\gamma e\cos(\psi+h+2g+3l)$$

(36) $$+\frac{17}{2}\gamma e^2\cos(\psi+h+2g+4l)$$

(37) $$-\frac{1}{2}\gamma e\cos(\psi+h+2g+l)$$

(38) $$+\left[-\frac{5}{8}\gamma e^2+\frac{15}{4}\gamma^3e^2-\frac{73}{96}\gamma e^4+\frac{135}{64}\gamma e^2\frac{n'}{n}+\frac{4757}{1024}\gamma e^2\frac{n'^2}{n^2}\right]\cos(\psi+h+2g)$$

(39) $$+\Big[-\gamma+\frac{5}{2}\gamma^3-\frac{3}{2}\gamma e^2-\frac{7}{8}\gamma^5+\frac{15}{4}\gamma^3e^2-\frac{215}{128}\gamma e^4$$
$$+\left(\frac{249}{128}\gamma-\frac{4217}{256}\gamma^3+\frac{1043}{256}\gamma e^2+\frac{535}{128}\gamma e'^2\right)\frac{n'^2}{n^2}-\frac{51}{256}\gamma\frac{n'^3}{n^3}-\frac{491867}{32768}\gamma\frac{n'^4}{n^4}\Big]$$
$$\times\cos(\psi+h)$$

(40) $$+\left[-\frac{9}{8}\gamma e'\frac{n'}{n}+\frac{213}{64}\gamma e'\frac{n'^2}{n^2}\right]\cos(\psi+h-l')$$

(41) $$-\frac{27}{32}\gamma e'^2\frac{n'}{n}\cos(\psi+h-2l')$$

(42) $$+\left[\frac{9}{8}\gamma e'\frac{n'}{n}+\frac{135}{64}\gamma e'\frac{n'^2}{n^2}\right]\cos(\psi+h+l')$$

(43) $$+\frac{27}{32}\gamma e'^2\frac{n'}{n}\cos(\psi+h+2l')$$

(44) $$-\frac{3}{2}\gamma e\cos(\psi+h+l)$$

(45) $$-\frac{13}{8}\gamma e^2\cos(\psi+h+2l)$$

(46) $$-\frac{3}{2}\gamma e\cos(\psi+h-l)$$

(47) $$-\frac{9}{4}\gamma e^2\cos(\psi+h-2l)$$

(48) $$-\gamma^3\cos(\psi+h-2g-2l)$$

(49) $$+\frac{115}{16}\gamma^3e^2\cos(\psi+h-2g)$$

(50) $$+\frac{23}{8}\gamma\frac{n'^2}{n^2}\cos(\psi+3h+4g+4l-2h'-2g'-2l')$$

(51) $$+\frac{105}{16}\gamma e\frac{n'}{n}\cos(\psi+3h+4g+3l-2h'-2g'-2l')$$

(52) $$+\left[\frac{3}{8}\gamma\frac{n'}{n}-\frac{45}{32}\gamma\frac{n'^2}{n^2}\right]\cos(\psi+3h+2g+2l-2h'-2g'-2l')$$

(7) $+\frac{3}{2}e^2\cos 2l$

(8) $+\frac{53}{24}e^3\cos 3l$

(9) $+2\gamma^2\cos(2g+2l)$

(10) $+7\gamma^2 e\cos(2g+3l)$

(11) $-\frac{7}{2}\gamma^2 e\cos(2g+l)$

(12) $+\left[-5\gamma^2e^2+\frac{135}{8}\gamma^2e^2\frac{n'}{n}\right]\cos 2g$

(13) $+\left[\left(\frac{3}{2}\gamma^2+\frac{45}{8}e^2\right)\frac{n'}{n}+\frac{n'^2}{n^2}+\frac{19}{6}\frac{n'^3}{n^3}\right]\cos(2h+2g+2l-2h'-2g'-2l')$

(14) $+\frac{7}{2}e'\frac{n'^2}{n^2}\cos(2h+2g+2l-2h'-2g'-3l')$

(15) $-\frac{1}{2}e'\frac{n'^2}{n^2}\cos(2h+2g+2l-2h'-2g'-l')$

(16) $+\frac{49}{16}e\frac{n'^2}{n^2}\cos(2h+2g+3l-2h'-2g'-2l')$

(17) $+\left[\frac{15}{8}e\frac{n'}{n}+\frac{219}{32}e\frac{n'^2}{n^2}\right]\cos(2h+2g+l-2h'-2g'-2l')$

(18) $+\frac{35}{8}ee'\frac{n'}{n}\cos(2h+2g+l-2h'-2g'-3l')$

(19) $-\frac{15}{8}ee'\frac{n'}{n}\cos(2h+2g+l-2h'-2g'-l')$

(20) $+\left[\frac{15}{8}e^2\frac{n'}{n}+\frac{107}{32}e^2\frac{n'^2}{n^2}\right]\cos(2h+2g-2h'-2g'-2l')$

(21) $+\frac{35}{8}e^2e'\frac{n'}{n}\cos(2h+2g-2h'-2g'-3l')$

(22) $-\frac{15}{8}e^2e'\frac{n'}{n}\cos(2h+2g-2h'-2g'-l')$

(23) $-\frac{45}{32}e^2e'^2\frac{n'}{n}\cos(2h+2g-2h'-2g')$

(24) $-\left[\frac{3}{2}\gamma^2\frac{n'}{n}+\frac{25}{8}\gamma^2\frac{n'^2}{n^2}\right]\cos(2h-2h'-2g'-2l')$

(25) $-\frac{7}{2}\gamma^2e'\frac{n'}{n}\cos(2h-2h'-2g'-3l')$

(26) $+\frac{3}{2}\gamma^2e'\frac{n'}{n}\cos(2h-2h'-2g'-l')$

(27) $+\frac{9}{8}\gamma^2e'^2\frac{n'}{n}\cos(2h-2h'-2g')$

(28) $-\frac{15}{16}\frac{n'}{n}\frac{a}{a'}\cos(h+g+l-h'-g'-l')$

(29) $+\frac{5}{4}e'\frac{a}{a'}\cos(h+g+l-h'-g')$

(30) $-\frac{15}{16}e\frac{n'}{n}\frac{a}{a'}\cos(h+g-h'-g'-l')$

(31) $$+\left[\frac{5}{4}ee'\frac{a}{a'} - \frac{45}{8}ee'\frac{n'}{n}\frac{a}{a'}\right]\cos(h + g - h' - g')\Big\}$$

(32) $$+\beta_2 n^2\Big\{\left[\gamma - \frac{3}{2}\gamma^3 - \frac{5}{2}\gamma e^2 - \frac{249}{128}\gamma\frac{n'^2}{n^2}\right]\cos(\psi + h + 2g + 2l)$$

(33) $$+\frac{15}{8}\gamma e'\frac{n'}{n}\cos(\psi + h + 2g + 2l - l')$$

(34) $$-\frac{15}{8}\gamma e'\frac{n'}{n}\cos(\psi + h + 2g + 2l + l')$$

(35) $$+\frac{7}{2}\gamma e\cos(\psi + h + 2g + 3l)$$

(36) $$+\frac{17}{2}\gamma e^2\cos(\psi + h + 2g + 4l)$$

(37) $$-\frac{1}{2}\gamma e\cos(\psi + h + 2g + l)$$

(38) $$+\left[-\frac{5}{8}\gamma e^2 + \frac{15}{4}\gamma^3 e^2 - \frac{73}{96}\gamma e^4 + \frac{135}{64}\gamma e^2\frac{n'}{n} + \frac{4757}{1024}\gamma e^2\frac{n'^2}{n^2}\right]\cos(\psi + h + 2g)$$

(39) $$+\Big[-\gamma + \frac{5}{2}\gamma^3 - \frac{3}{2}\gamma e^2 - \frac{7}{8}\gamma^5 + \frac{15}{4}\gamma^3 e^2 - \frac{215}{128}\gamma e^4$$
$$+\left(\frac{249}{128}\gamma - \frac{4217}{256}\gamma^3 + \frac{1043}{256}\gamma e^2 + \frac{535}{128}\gamma e'^2\right)\frac{n'^2}{n^2} - \frac{51}{256}\gamma\frac{n'^3}{n^3} - \frac{491867}{32768}\gamma\frac{n'^4}{n^4}\Big]$$
$$\times\cos(\psi + h)$$

(40) $$+\left[-\frac{9}{8}\gamma e'\frac{n'}{n} + \frac{213}{64}\gamma e'\frac{n'^2}{n^2}\right]\cos(\psi + h - l')$$

(41) $$-\frac{27}{32}\gamma e'^2\frac{n'}{n}\cos(\psi + h - 2l')$$

(42) $$+\left[\frac{9}{8}\gamma e'\frac{n'}{n} + \frac{135}{64}\gamma e'\frac{n'^2}{n^2}\right]\cos(\psi + h + l')$$

(43) $$+\frac{27}{32}\gamma e'^2\frac{n'}{n}\cos(\psi + h + 2l')$$

(44) $$-\frac{3}{2}\gamma e\cos(\psi + h + l)$$

(45) $$-\frac{13}{8}\gamma e^2\cos(\psi + h + 2l)$$

(46) $$-\frac{3}{2}\gamma e\cos(\psi + h - l)$$

(47) $$-\frac{9}{4}\gamma e^2\cos(\psi + h - 2l)$$

(48) $$-\gamma^3\cos(\psi + h - 2g - 2l)$$

(49) $$+\frac{115}{16}\gamma^3 e^2\cos(\psi + h - 2g)$$

(50) $$+\frac{23}{8}\gamma\frac{n'^2}{n^2}\cos(\psi + 3h + 4g + 4l - 2h' - 2g' - 2l')$$

(51) $$+\frac{105}{16}\gamma e\frac{n'}{n}\cos(\psi + 3h + 4g + 3l - 2h' - 2g' - 2l')$$

(52) $$+\left[\frac{3}{8}\gamma\frac{n'}{n} - \frac{45}{32}\gamma\frac{n'^2}{n^2}\right]\cos(\psi + 3h + 2g + 2l - 2h' - 2g' - 2l')$$

(53) $+\frac{7}{8}\gamma e'\frac{n'}{n}\cos(\psi+3h+2g+2l-2h'-2g'-3l')$

(54) $-\frac{3}{8}\gamma e'\frac{n'}{n}\cos(\psi+3h+2g+2l-2h'-2g'-l')$

(55) $+\frac{21}{16}\gamma e\frac{n'}{n}\cos(\psi+3h+2g+3l-2h'-2g'-2l')$

(56) $-3\gamma e\frac{n'}{n}\cos(\psi+3h+2g+l-2h'-2g'-2l')$

(57) $-\frac{15}{4}\gamma e^2\frac{n'}{n}\cos(\psi+3h+2g-2h'-2g'-2l')$

(58) $+\frac{3}{4}\gamma^3\frac{n'}{n}\cos(\psi+3h-2h'-2g'-2l')$

(59) $+\left[\left(-\frac{3}{8}\gamma+\frac{39}{16}\gamma^3-\frac{21}{16}\gamma e^2+\frac{15}{16}\gamma e'^2\right)\frac{n'}{n}+\frac{1}{32}\gamma\frac{n'^2}{n^2}+\frac{2215}{3072}\gamma\frac{n'^3}{n^3}\right]$
$\times\cos(\psi-h+2h'+2g'+2l')$

(60) $+\frac{77}{64}\gamma e'\frac{n'^2}{n^2}\cos(\psi-h+2h'+2g'+l')$

(61) $+\left[\frac{9}{32}\gamma e'^2\frac{n'}{n}+\frac{93}{256}\gamma e'^2\frac{n'^2}{n^2}\right]\cos(\psi-h+2h'+2g')$

(62) $+\left[-\frac{7}{8}\gamma e'\frac{n'}{n}-\frac{37}{64}\gamma e'\frac{n'^2}{n^2}\right]\cos(\psi-h+2h'+2g'+3l')$

(63) $-\frac{51}{32}\gamma e'^2\frac{n'}{n}\cos(\psi-h+2h'+2g'+4l')$

(64) $-\frac{3}{2}\gamma e\frac{n'}{n}\cos(\psi-h+l+2h'+2g'+2l')$

(65) $-\frac{9}{16}\gamma e\frac{n'}{n}\cos(\psi-h-l+2h'+2g'+2l')$

(66) $-\frac{3}{2}\gamma\frac{n'^2}{n^2}\cos(\psi-h-2g-2l+2h'+2g'+2l')$

(67) $-\frac{45}{16}\gamma e\frac{n'}{n}\cos(\psi-h-2g-l+2h'+2g'+2l')$

(68) $-\frac{195}{64}\gamma e^2\frac{n'}{n}\cos(\psi-h-2g+2h'+2g'+2l')$

(69) $-\left[\frac{25}{8}\gamma ee'\frac{a}{a'}+\frac{5}{16}\gamma ee'\frac{n'}{n}\frac{a}{a'}\right]\cos(\psi+2h+g-h'-g')$

(70) $-\frac{5}{6}\gamma ee'\frac{a}{a'}\cos(\psi+g+h'+g')$

(71) $-\frac{5}{4}\gamma ee'\frac{a}{a'}\cos(\psi-g+h'+g')\Big\}$

(72) $+\beta_3 n^2\Big\{\left[\frac{1}{2}-\gamma^2-\frac{5}{4}e^2-\frac{3}{4}\frac{n'^2}{n^2}\right]\cos(2\psi+2h+2g+2l)$

(73) $+\left[\frac{3}{2}e'\frac{n'}{n}-\frac{9}{8}e'\frac{n'^2}{n^2}\right]\cos(2\psi+2h+2g+2l-l')$

(74) $+\frac{9}{8}e'^2\frac{n'}{n}\cos(2\psi+2h+2g+2l-2l')$

(75) $-\left[\frac{3}{2}e'\frac{n'}{n}+\frac{9}{8}e'\frac{n'^2}{n^2}\right]\cos(2\psi+2h+2g+2l+l')$

(76) $$-\frac{9}{8}e'^2\frac{n'}{n}\cos(2\psi + 2h + 2g + 2l + 2l')$$

(77) $$+\left[\frac{7}{4}e - \frac{7}{2}\gamma^2 e - \frac{123}{32}e^3 - \frac{2199}{512}e\frac{n'^2}{n^2}\right]\cos(2\psi + 2h + 2g + 3l)$$

(78) $$+\frac{315}{32}ee'\frac{n'}{n}\cos(2\psi + 2h + 2g + 3l - l')$$

(79) $$-\frac{315}{32}ee'\frac{n'}{n}\cos(2\psi + 2h + 2g + 3l + l')$$

(80) $$+\frac{17}{4}e^2\cos(2\psi + 2h + 2g + 4l)$$

(81) $$+\frac{845}{96}e^3\cos(2\psi + 2h + 2g + 5l)$$

(82) $$+\left[-\frac{1}{4}e + \frac{1}{2}\gamma^2 e + \frac{1}{32}e^3 - \frac{15}{512}e\frac{n'^2}{n^2}\right]\cos(2\psi + 2h + 2g + l)$$

(83) $$-\frac{3}{32}ee'\frac{n'}{n}\cos(2\psi + 2h + 2g + l - l')$$

(84) $$+\frac{3}{32}ee'\frac{n'}{n}\cos(2\psi + 2h + 2g + l + l')$$

(85) $$+\left[\frac{5}{4}\gamma^2 e^2 - \frac{135}{32}\gamma^2 e^2\frac{n'}{n} + \frac{1}{32}e^2\frac{n'^2}{n^2} - \frac{225}{32}e^2\frac{n'^3}{n^3}\right]\cos(2\psi + 2h + 2g)$$

(88) $$+\frac{1}{96}e^3\cos(2\psi + 2h + 2g - l)$$

(89) $$-\frac{35}{8}\gamma^2 e\cos(2\psi + 2h + 4g + 3l)$$

(90) $$+\left[\gamma^2 - \gamma^4 + \frac{3}{2}\gamma^2 e^2 - \frac{145}{64}\gamma^2\frac{n'^2}{n^2} + \frac{51}{128}\gamma^2\frac{n'^3}{n^3}\right]\cos(2\psi + 2h)$$

(91) $$+\frac{9}{4}\gamma^2 e'\frac{n'}{n}\cos(2\psi + 2h - l')$$

(92) $$-\frac{9}{4}\gamma^2 e'\frac{n'}{n}\cos(2\psi + 2h + l')$$

(93) $$+\frac{17}{8}\gamma^2 e\cos(2\psi + 2h + l)$$

(94) $$+\frac{3}{2}\gamma^2 e\cos(2\psi + 2h - l)$$

(95) $$+\left[\frac{255}{16}e^2\frac{n'}{n} + \frac{23}{16}\frac{n'^2}{n^2} + \frac{29}{6}\frac{n'^3}{n^3}\right]\cos(2\psi + 4h + 4g + 4l - 2h' - 2g' - 2l')$$

(96) $$+\frac{161}{32}e'\frac{n'^2}{n^2}\cos(2\psi + 4h + 4g + 4l - 2h' - 2g' - 3l')$$

(97) $$-\frac{23}{32}e'\frac{n'^2}{n^2}\cos(2\psi + 4h + 4g + 4l - 2h' - 2g' - l')$$

(98) $$+\frac{465}{64}e\frac{n'^2}{n^2}\cos(2\psi + 4h + 4g + 5l - 2h' - 2g' - 2l')$$

(99) $$+\left[\frac{105}{32}e\frac{n'}{n} + \frac{1473}{128}e\frac{n'^2}{n^2}\right]\cos(2\psi + 4h + 4g + 3l - 2h' - 2g' - 2l')$$

(100) $$+\frac{245}{32}ee'\frac{n'}{n}\cos(2\psi + 4h + 4g + 3l - 2h' - 2g' - 3l')$$

(101) $$-\frac{105}{32}ee'\frac{n'}{n}\cos(2\psi+4h+4g+3l-2h'-2g'-l')$$

(102) $$-\frac{15}{16}e^2\frac{n'}{n}\cos(2\psi+4h+4g+2l-2h'-2g'-2l')$$

(103) $$+\frac{3}{4}\gamma^2\frac{n'}{n}\cos(2\psi+4h+2g+2l-2h'-2g'-2l')$$

(104) $$+\left[\frac{3}{4}\gamma^2\frac{n'}{n}+\left(\frac{1}{16}-\frac{1}{16}\gamma^2-\frac{5}{32}e'^2\right)\frac{n'^2}{n^2}-\frac{1}{12}\frac{n'^3}{n^3}-\frac{241}{288}\frac{n'^4}{n^4}\right] \times\cos(2\psi+2h'+2g'+2l')$$

(105) $$+\left[-\frac{3}{4}\gamma^2e'\frac{n'}{n}-\frac{1}{32}e'\frac{n'^2}{n^2}+\frac{1}{48}e'\frac{n'^3}{n^3}\right]\cos(2\psi+2h'+2g'+l')$$

(106) $$-\frac{9}{16}\gamma^2e'^2\frac{n'}{n}\cos(2\psi+2h'+2g')$$

(107) $$+\left[\frac{7}{4}\gamma^2e'\frac{n'}{n}+\frac{7}{32}e'\frac{n'^2}{n^2}-\frac{7}{16}e'\frac{n'^3}{n^3}\right]\cos(2\psi+2h'+2g'+3l')$$

(108) $$+\frac{17}{32}e'^2\frac{n'^2}{n^2}\cos(2\psi+2h'+2g'+4l')$$

(109) $$+\left[-\frac{15}{32}e\frac{n'}{n}-\frac{511}{128}e\frac{n'^2}{n^2}\right]\cos(2\psi+l+2h'+2g'+2l')$$

(110) $$+\frac{15}{32}ee'\frac{n'}{n}\cos(2\psi+l+2h'+2g'+l')$$

(111) $$-\frac{35}{32}ee'\frac{n'}{n}\cos(2\psi+l+2h'+2g'+3l')$$

(112) $$-\frac{15}{4}e^2\frac{n'}{n}\cos(2\psi+2l+2h'+2g'+2l')$$

(113) $$+\frac{5}{64}e\frac{n'^2}{n^2}\cos(2\psi-l+2h'+2g'+2l')$$

(114) $$-\frac{3}{2}\gamma^2\frac{n'}{n}\cos(2\psi+2g+2l+2h'+2g'+2l')$$

(115) $$-\frac{225}{64}e^2\frac{n'^2}{n^2}\cos(2\psi-2h-2g+4h'+4g'+4l')$$

(116) $$+\frac{9}{64}\gamma^2\frac{n'^2}{n^2}\cos(2\psi-2h+4h'+4g'+4l')$$

(117) $$-\frac{105}{64}\frac{n'}{n}\frac{a}{a'}\cos(2\psi+3h+3g+3l-h'-g'-l')$$

(118) $$+\frac{35}{16}e'\frac{a}{a'}\cos(2\psi+3h+3g+3l-h'-g')$$

(119) $$+\frac{15}{64}\frac{n'}{n}\frac{a}{a'}\cos(2\psi+h+g+l+h'+g'+l')$$

(120) $$-\frac{5}{16}e'\frac{a}{a'}\cos(2\psi+h+g+l+h'+g')$$

(122) $$-\frac{135}{16}ee'\frac{n'}{n}\frac{a}{a'}\cos(2\psi+h+g+h'+g')\Big\}.$$

CHAPTER II.

DETAIL OF THE OPERATIONS NECESSARY FOR REMOVING FROM THE PERTURBATIVE FUNCTION THE PERIODIC TERMS WHICH ARE PRODUCED BY THE FIGURE OF THE EARTH.

The differential equations, which the variables a, e, γ, l, g, and h satisfy, are

$$\frac{da}{dt} = \frac{2}{an}\frac{d\mathrm{R}}{dl} - \frac{1}{an}\left\{\frac{15}{16}\frac{n'^4}{n^4} + \frac{167}{8}\frac{n'^5}{n^5}\right\}\frac{d\mathrm{R}}{dh},$$

$$\begin{aligned}\frac{de}{dt} = {} & \frac{1}{a^2ne}\left\{1 - e^2 + \frac{225}{32}\frac{n'^2}{n^2} + \frac{675}{64}\frac{n'^3}{n^3}\right\}\frac{d\mathrm{R}}{dl} - \frac{1}{a^2ne}\left\{1 - \frac{1}{2}e^2 + \frac{225}{32}\frac{n'^2}{n^2} + \frac{675}{64}\frac{n'^3}{n^3}\right\}\frac{d\mathrm{R}}{dg} \\ & + \frac{1}{a^2ne}\left\{-\frac{25}{4}\gamma^2e^2 + \frac{25}{32}e^4 + \frac{225}{32}e^2\frac{n'^2}{n^2}\right\}\frac{d\mathrm{R}}{dh},\end{aligned}$$

$$\begin{aligned}\frac{d\gamma}{dt} = {} & \frac{1}{4a^2n\gamma}\left\{1 - 2\gamma^2 + \frac{1}{2}e^2 + \frac{9}{32}\frac{n'^2}{n^2} - \frac{27}{64}\frac{n'^3}{n^3}\right\}\frac{d\mathrm{R}}{dg} - \frac{1}{4a^2n\gamma}\left\{1 + \frac{1}{2}e^2 - \frac{25}{4}\gamma^2e^2 + \frac{37}{32}e^4\right. \\ & \left. + \left(\frac{9}{32} - \frac{27}{16}\gamma^2 + \frac{81}{64}e^2 + \frac{13}{32}e'^2\right)\frac{n'^2}{n^2} - \frac{27}{64}\frac{n'^3}{n^3} + \frac{5711}{2048}\frac{n'^4}{n^4}\right\}\frac{d\mathrm{R}}{dh},\end{aligned}$$

$$\begin{aligned}\frac{d(h+g+l)}{dt} = {} & n\left\{1 - \left(1 - \frac{9}{2}\gamma^2 + \frac{9}{8}e^2 + \frac{3}{2}e'^2 + 3\gamma^4 - \frac{15}{4}\gamma^2e^2 - \frac{27}{4}\gamma^2e'^2\right)\frac{n'^2}{n^2} - \left(\frac{27}{8}\gamma^2 + \frac{675}{32}e^2\right.\right. \\ & \left.\left. - \frac{135}{16}\gamma^4 - \frac{243}{4}\gamma^2e^2 + \frac{69}{8}\gamma^2e'^2\right)\frac{n'^3}{n^3} + \left(\frac{451}{64} - \frac{747}{32}\gamma^2\right)\frac{n'^4}{n^4} + \left(\frac{787}{32} - \frac{8043}{128}\gamma^2\right)\frac{n'^5}{n^5}\right\} \\ & - \frac{2}{an}\frac{d\mathrm{R}}{da} + \frac{1}{2}\frac{e}{a^2n}\frac{d\mathrm{R}}{de} + \frac{1}{a^2n\gamma}\left\{\frac{1}{2}\gamma^2 + \frac{1}{4}\gamma^2e^2 - \frac{27}{32}\gamma^2\frac{n'^2}{n^2} + \frac{243}{128}\gamma^2\frac{n'^3}{n^3}\right\}\frac{d\mathrm{R}}{d\gamma},\end{aligned}$$

$$\begin{aligned}\frac{dl}{dt} = {} & n\left\{1 - \left(\frac{7}{4} - \frac{21}{2}\gamma^2 + \frac{3}{4}e^2 + \frac{21}{8}e'^2 - \frac{33}{4}\gamma^4 + \frac{39}{8}\gamma^2e^2 - \frac{63}{4}\gamma^2e'^2\right)\frac{n'^2}{n^2}\right. \\ & \left. + \left(-\frac{225}{32} + \frac{81}{4}\gamma^2\right)\frac{n'^3}{n^3} + \left(-\frac{3265}{128} + \frac{3345}{32}\gamma^2\right)\frac{n'^4}{n^4}\right\} - \frac{2}{an}\frac{d\mathrm{R}}{da} - \\ & - \frac{1}{a^2ne}\left\{1 - e^2 + \frac{225}{32}\frac{n'^2}{n^2} + \frac{675}{64}\frac{n'^3}{n^3}\right\}\frac{d\mathrm{R}}{de} - \frac{1}{a^2n\gamma}\left\{-\frac{25}{8}\gamma^4 + \frac{25}{16}\gamma^2e^2 + \frac{351}{64}\gamma^2\frac{n'^2}{n^2}\right\}\frac{d\mathrm{R}}{d\gamma},\end{aligned}$$

$$\begin{aligned}\frac{dh}{dt} = {} & -n\left\{\left(\frac{3}{4} - \frac{3}{2}\gamma^2 + \frac{3}{2}e^2 + \frac{9}{8}e'^2 + \frac{51}{8}\gamma^2e^2 - \frac{9}{4}\gamma^2e'^2 - \frac{21}{64}e^4 + \frac{9}{4}e^2e'^2 + \frac{45}{32}e'^4\right)\frac{n'^2}{n^2}\right. \\ & - \left(\frac{9}{32} - \frac{27}{16}\gamma^2 - \frac{189}{32}e^2 + \frac{23}{32}e'^2\right)\frac{n'^3}{n^3} - \left(\frac{177}{128} - \frac{195}{64}\gamma^2 - \frac{699}{32}e^2 + \frac{2685}{256}e'^2\right)\frac{n'^4}{n^4} \\ & \left. - \frac{10949}{2048}\frac{n'^5}{n^5} - \frac{467977}{24576}\frac{n'^6}{n^6} + \frac{45}{32}\frac{n'^2}{n^2}\frac{a^2}{a'^2}\right\} + \frac{1}{4a^2n\gamma}\left\{1 + \frac{1}{2}e^2 - \frac{25}{4}\gamma^2e^2 + \frac{37}{32}e^4\right. \\ & \left. + \left(\frac{9}{32} - \frac{27}{16}\gamma^2 + \frac{81}{64}e^2 + \frac{13}{32}e'^2\right)\frac{n'^2}{n^2} - \frac{27}{64}\frac{n'^3}{n^3} + \frac{5711}{2048}\frac{n'^4}{n^4}\right\}\frac{d\mathrm{R}}{d\gamma}.\end{aligned}$$

This great extent in the differential equations is required in only one of the following operations, viz, the 32d. In general much shorter forms of them suffice. The value of the partial derivatives $\frac{da}{dL}, \frac{da}{dG}, \frac{da}{dH}, \frac{de}{dL}, \frac{de}{dG}, \frac{de}{dH}$ will be found in DELAUNAY, Tom. I, pp. 834, 835. Those of the derivatives $\frac{d\gamma}{dL}, \frac{d\gamma}{dG}, \frac{d\gamma}{dH}$, Tom. I, pp. 857, 858. The portions of $\frac{dl}{dt}$ and $\frac{dh}{dt}$, which are independent of the partial derivatives, are given, Tom. II, pp. 237, 238; and the similar portion of $\frac{d(h+g+l)}{dt}$, Tom. II, p. 799.

In integrating we generally disregard the motion of ψ. But in two operations, viz, the 32d and 79th, our convention of retaining all quantities to the seventh order, inclusive, demands that we take it into account. For this purpose we denote it as ft, and call $\frac{f}{n}$ a quantity of the fifth order.

In taking the partial derivatives it must always be borne in mind that n is only an abbreviation for $\frac{\sqrt{\mu}}{a\sqrt{a}}$. In each of the operations we find, first, the values of the augmentations of a, e, and γ from the first three differential equations, and, afterwards, having obtained the corresponding augmentations of the terms of the right members of the three last equations, which are independent of the partial derivatives of R, we add to them what results from the terms which involve these partial derivatives. And thus, after integration, we have the proper augmentations of $h+g+l$, l, and h.

After the integration, we make the same transformation in the expressions of the coefficients as DELAUNAY has given (Tom. II, p 800), and which is the reverse of that we made before giving the final development of R; that is, we replace

$$a \text{ by } a\left\{1 - \left(\frac{2}{3} - 3\gamma^2 + \frac{3}{4}e^2 + e'^2\right)\frac{n'^2}{n^2} - \left(\frac{9}{4}\gamma^2 + \frac{225}{16}e^2\right)\frac{n'^3}{n^3} + \frac{1705}{288}\frac{n'^4}{n^4} + \frac{787}{48}\frac{n'^5}{n^5}\right\},$$

$$e \text{ by } e\left\{1 + \frac{81}{128}\frac{n'^2}{n^2} - \frac{2595}{256}\frac{n'^3}{n^3}\right\},$$

$$\gamma \text{ by } \gamma\left\{1 + \frac{5}{8}\gamma^2 e^2 + \frac{15}{128}e^4 + \left(\frac{57}{128} - \frac{293}{128}\gamma^2 + \frac{991}{256}e^2 + \frac{103}{128}e'^2\right)\frac{n'^2}{n^2} - \frac{129}{256}\frac{n'^3}{n^3} - \frac{22457}{32768}\frac{n'^4}{n^4}\right\}.$$

Consequently the transformations we give in the detailed operations, which follow, are directly applicable to DELAUNAY'S expressions of V, U, and $\frac{a}{r}$, which are given, Tom. II, pp. 803–924. It will be perceived that this transformation affects only the operations which are numbered (1), (25), (31), (32), (52), (65), (68), (73), (79).

Operation 1.—*Term* (4) *of* R.

We replace

$$a \text{ by } a\left\{1 + 2\frac{\beta_1}{a^2}e\cos l\right\},$$

$$e \text{ by } e + \frac{\beta_1}{a^2}\left[1 - 6\gamma^2 + \frac{1}{8}e^2 + \frac{1267}{192}m^2\right]\cos l,$$

$$h+g+l \text{ by } h+g+l+\frac{7}{2}\frac{\beta_1}{a^2}e\sin l,$$

$$l \text{ by } l - \frac{\beta_1}{a^2}\left[1 - 6\gamma^2 - \frac{5}{8}e^2 + \frac{1267}{192}m^2\right]\frac{1}{e}\sin l,$$

$$h \text{ by } h - 3\frac{\beta_1}{a^2}e\sin l,$$

γ does not change.

Operation 2.—*Term* (5) *of* R.

We replace

$$e \text{ by } e + \frac{21}{8}\frac{\beta_1}{a^2}e'm\cos(l - l'),$$

$$l \text{ by } l - \frac{21}{8}\frac{\beta_1}{a^2}\frac{e'm}{e}\sin(l - l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 3.—*Term* (6) *of* R.

We replace

$$e \text{ by } e - \frac{21}{8}\frac{\beta_1}{a^2}e'm\cos(l + l'),$$

$$l \text{ by } l + \frac{21}{8}\frac{\beta_1}{a^2}\frac{e'm}{e}\sin(l + l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 4.—*Term* (7) *of* R.

We replace

$$a \text{ by } a\left\{1 + 3\frac{\beta_1}{a^2}e^2\cos 2l\right\}$$

$$e \text{ by } e + \frac{3}{2}\frac{\beta_1}{a^2}e\cos 2l,$$

$$h+g+l \text{ by } h+g+l + 3\frac{\beta_1}{a^2}e^2\sin 2l,$$

$$l \text{ by } l - \frac{3}{2}\frac{\beta_1}{a^2}\sin 2l.$$

γ and h do not change.

Operation 5.—*Term* (8) *of* R.

We replace

$$e \text{ by } e + \frac{53}{24}\frac{\beta_1}{a^2}e^2\cos 3l,$$

$$l \text{ by } l - \frac{53}{24}\frac{\beta_1}{a^2}e\sin 3l,$$

a, γ, $h+g+l$, and h do not change.

Operation 6.—*Term* (9) *of* R.

We replace

$$a \text{ by } a\left\{1 + 4\frac{\beta_1}{a^2}\gamma^2\cos(2g + 2l)\right\},$$

$$\gamma \text{ by } \gamma + \frac{1}{2}\frac{\beta_1}{a^2}\gamma\cos(2g + 2l),$$

$$h+g+l \text{ by } h+g+l + 4\frac{\beta_1}{a^2}\gamma^2\sin(2g + 2l),$$

$$h \text{ by } h + \frac{1}{2}\frac{\beta_1}{a^2}\sin(2g + 2l),$$

e and l do not change.

Operation 7.—*Term* (10) *of* R.

We replace

$$e \text{ by } e + \frac{7}{3}\frac{\beta_1}{a^3}\gamma^2 \cos(2g+3l),$$

$$\gamma \text{ by } \gamma + \frac{7}{6}\frac{\beta_1}{a^3}\gamma e \cos(2g+3l),$$

$$l \text{ by } l - \frac{7}{3}\frac{\beta_1}{a^3}\frac{\gamma^2}{e} \sin(2g+3l),$$

$$h \text{ by } h + \frac{7}{6}\frac{\beta_1}{a^3} e \sin(2g+3l),$$

a and $h+g+l$ do not change.

Operation 8.—*Term* (11) *of* R.

We replace

$$e \text{ by } e + \frac{7}{2}\frac{\beta_1}{a^3}\gamma^2 \cos(2g+l),$$

$$\gamma \text{ by } \gamma - \frac{7}{4}\frac{\beta_1}{a^3}\gamma e \cos(2g+l),$$

$$l \text{ by } l + \frac{7}{2}\frac{\beta_1}{a^3}\frac{\gamma^2}{e} \sin(2g+l),$$

$$h \text{ by } h - \frac{7}{4}\frac{\beta_1}{a^3} e \sin(2g+l),$$

a and $h+g+l$ do not change.

Operation 9.—*Term* (12) *of* R.

We replace

$$e \text{ by } e + \frac{\beta_1}{a^3 m^3}\left[\frac{10}{3}\gamma^2 e - \frac{105}{4}\gamma^2 e m\right]\cos 2g,$$

$$\gamma \text{ by } \gamma - \frac{\beta_1}{a^3 m^3}\left[\frac{5}{6}\gamma e^2 - \frac{105}{16}\gamma e^2 m\right]\cos 2g,$$

$$h+g+l \text{ by } h+g+l - \frac{55}{3}\frac{\beta_1}{a^3 m^3}\gamma^2 e^2 \sin 2g,$$

$$l \text{ by } l + \frac{\beta_1}{a^3 m^3}\left[\frac{10}{3}\gamma^2 - \frac{105}{4}\gamma^2 m\right]\sin 2g,$$

$$h \text{ by } h - \frac{\beta_1}{a^3 m^3}\left[\frac{5}{6}e^2 - \frac{105}{16}e^2 m\right]\sin 2g,$$

a does not change.

Operation 10.—*Term* (13) *of* R.

We replace

$$a \text{ by } a\left\{1 + 2\frac{\beta_1}{a^3}m^2\cos(2h+2g+2l-2h'-2g'-2l')\right\},$$

$$h+g+l \text{ by } h+g+l - \frac{3}{2}\frac{\beta_1}{a^3}m^2 \sin(2h+2g+2l-2h'-2g'-2l'),$$

$$l \text{ by } l - \frac{45}{8}\frac{\beta_1}{a^3} m \sin(2h+2g+2l-2h'-2g'-2l'),$$

$$h \text{ by } h + \frac{3}{8}\frac{\beta_1}{a^3} m \sin(2h+2g+2l-2h'-2g'-2l'),$$

e and γ do not change.

Operation 11.—*Term* (16) *of* R.

We replace

$$e \text{ by } e + \frac{49}{48}\frac{\beta_1}{a^2}m^2 \cos(2h + 2g + 3l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{49}{48}\frac{\beta_1}{a^2}m^2\frac{1}{e}\sin(2h + 2g + 3l - 2h' - 2g' - 2l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 12.—*Term* (17) *of* R.

We replace

$$a \text{ by } a\left\{1 + \frac{15}{4}\frac{\beta_1}{a^2}em\cos(2h + 2g + l - 2h' - 2g' - 2l')\right\},$$

$$e \text{ by } e - \frac{\beta_1}{a^2}\left[\frac{15}{8}m + \frac{339}{32}m^2\right]\cos(2h + 2g + l - 2h' - 2g' - 2l'),$$

$$h + g + l \text{ by } h + g + l + \frac{15}{16}\frac{\beta_1}{a^2}em\sin(2h + 2g + l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{\beta_1}{a^2}\left[\frac{15}{8}m + \frac{339}{32}m^2\right]\frac{1}{e}\sin(2h + 2g + l - 2h' - 2g' - 2l').$$

γ and h do not change.

Operation 13.—*Term* (18) *of* R.

We replace

$$e \text{ by } e - \frac{35}{8}\frac{\beta_1}{a^2}e'm\cos(2h + 2g + l - 2h' - 2g' - 3l'),$$

$$l \text{ by } l - \frac{35}{8}\frac{\beta_1}{a^2}e'm\frac{1}{e}\sin(2h + 2g + l - 2h' - 2g' - 3l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 14.—*Term* (19) *of* R.

We replace

$$e \text{ by } e + \frac{15}{8}\frac{\beta_1}{a^2}e'm\cos(2h + 2g + l - 2h' - 2g' - l'),$$

$$l \text{ by } l + \frac{15}{8}\frac{\beta_1}{a^2}e'm\frac{1}{e}\sin(2h + 2g + l - 2h' - 2g' - l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 15.—*Term* (20) *of* R.

We replace

$$e \text{ by } e + \frac{\beta_1}{a^2}\left[\frac{15}{8}e + \frac{19}{4}em\right]\cos(2h + 2g - 2h' - 2g' - 2l'),$$

$$h + g + l \text{ by } h + g + l - \frac{15}{4}\frac{\beta_1}{a^2}e^2\sin(2h + 2g - 2h' - 2g' - 2l'),$$

$$l \text{ by } l + \frac{\beta_1}{a^2}\left[\frac{15}{8} + \frac{19}{4}m\right]\sin(2h + 2g - 2h' - 2g' - 2l'),$$

a, γ, and h do not change.

Operation 16.—*Term* (21) *of* R.

We replace

$$e \text{ by } e + \frac{35}{12}\frac{\beta_1}{a^3}ee' \cos(2h + 2g - 2h' - 2g' - 3l'),$$

$$l \text{ by } l + \frac{35}{12}\frac{\beta_1}{a^3}e' \sin(2h + 2g - 2h' - 2g' - 3l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 17.—*Term* (22) *of* R.

We replace

$$e \text{ by } e - \frac{15}{4}\frac{\beta_1}{a^3}ee' \cos(2h + 2g - 2h' - 2g' - l'),$$

$$l \text{ by } l - \frac{15}{4}\frac{\beta_1}{a^3}e' \sin(2h + 2g - 2h' - 2g' - l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 18.—*Term* (23) *of* R.

We replace

$$e \text{ by } e + \frac{15}{8}\frac{\beta_1}{a^3}\frac{ee'^2}{m} \cos(2h + 2g - 2h' - 2g'),$$

$$l \text{ by } l + \frac{15}{8}\frac{\beta_1}{a^3}\frac{e'^2}{m} \sin(2h + 2g - 2h' - 2g'),$$

a, γ, $h+g+l$, and h do not change.

Operation 19.—*Term* (24) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{\beta_1}{a^3}\left[\frac{3}{8}\gamma + \frac{1}{2}\gamma m\right] \cos(2h - 2h' - 2g' - 2l'),$$

$$h + g + l \text{ by } h + g + l + 3\frac{\beta_1}{a^3}\gamma^2 \sin(2h - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{\beta_1}{a^3}\left[\frac{3}{8} + \frac{1}{2}m\right] \sin(2h - 2h' - 2g' - 2l'),$$

a, e, and l do not change.

Operation 20.—*Term* (25) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{7}{12}\frac{\beta_1}{a^3}\gamma e' \cos(2h - 2h' - 2g' - 3l'),$$

$$h \text{ by } h + \frac{7}{12}\frac{\beta_1}{a^3}e' \sin(2h - 2h' - 2g' - 3l'),$$

a, e, $h+g+l$, and l do not change.

Operation 21.—*Term* (26) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{3}{4}\frac{\beta_1}{a^3}\gamma e' \cos(2h - 2h' - 2g' - l'),$$

$$h \text{ by } h - \frac{3}{4}\frac{\beta_1}{a^3}e' \sin(2h - 2h' - 2g' - l'),$$

a, e, $h+g+l$, and l do not change.

Operation 22.—*Term* (27) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{3}{8}\frac{\beta_1}{a^3}\frac{\gamma e'^2}{m}\cos(2h - 2h' - 2g'),$$

$$h \text{ by } h - \frac{3}{8}\frac{\beta_1}{a^3}\frac{e'^2}{m}\sin(2h - 2h' - 2g'),$$

a, e, $h+g+l$, and l do not change.

Operation 23.—*Term* (30) *of* R.

We replace

$$e \text{ by } e - \frac{15}{16}\frac{\beta_1}{a^3}\frac{a}{a'}\cos(h + g - h' - g' - l'),$$

$$l \text{ by } l - \frac{15}{16}\frac{\beta_1}{a^3}\frac{a}{a'}\frac{1}{e}\sin(h + g - h' - g' - l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 24.—*Term* (31) *of* R.

We replace

$$e \text{ by } e - \frac{\beta_1}{a^3}\left[\frac{5}{3}e' - \frac{185}{8}e'm\right]\frac{a}{a'}\frac{1}{m^2}\cos(h + g - h' - g'),$$

$$h+g+l \text{ by } h+g+l + \frac{25}{2}\frac{\beta_1}{a^3m^2}ee'\frac{a}{a'}\sin(h + g - h' - g'),$$

$$l \text{ by } l - \frac{\beta_1}{a^3m^2}\left[\frac{5}{3}e' - \frac{185}{8}e'm\right]\frac{a}{a'}\frac{1}{e}\sin(h + g - h' - g'),$$

a, γ, and h do not change.

Operation 25.—*Term* (32) *of* R.

We replace

$$a \text{ by } a\left\{1 + 2\frac{\beta_2}{a^3}\gamma\cos(\psi + h + 2g + 2l)\right\},$$

$$e \text{ by } e - \frac{1}{2}\frac{\beta_2}{a^3}\gamma e\cos(\psi + h + 2g + 2l),$$

$$\gamma \text{ by } \gamma + \frac{\beta_2}{a^3}\left[\frac{1}{8} - \frac{11}{16}\gamma^2 - \frac{1}{4}e^2 - \frac{29}{1536}m^2\right]\cos(\psi + h + 2g + 2l),$$

$$h+g+l \text{ by } h+g+l + \frac{7}{4}\frac{\beta_2}{a^3}\gamma\sin(\psi + h + 2g + 2l),$$

$$l \text{ by } l + 4\frac{\beta_2}{a^3}\gamma\sin(\psi + h + 2g + 2l),$$

$$h \text{ by } h + \frac{\beta_2}{a^3}\left[\frac{1}{8} - \frac{9}{16}\gamma^2 - \frac{1}{4}e^2 - \frac{29}{1536}m^2\right]\frac{1}{\gamma}\sin(\psi + h + 2g + 2l).$$

Operation 26.—*Term* (33) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{15}{64}\frac{\beta_2}{a^3}e'm\cos(\psi + h + 2g + 2l - l'),$$

$$h \text{ by } h + \frac{15}{64}\frac{\beta_2}{a^3}e'm\frac{1}{\gamma}\sin(\psi + h + 2g + 2l - l'),$$

a, e, $h+g+l$, and l do not change.

Operation 27.—*Term* (34) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{15}{64}\frac{\beta_2}{a^2}e'm\cos(\psi + h + 2g + 2l + l'),$$

$$h \text{ by } h - \frac{15}{64}\frac{\beta_2}{a^2}e'm\frac{1}{\gamma}\sin(\psi + h + 2g + 2l + l'),$$

a, e, $h+g+l$, and l do not change.

Operation 28.—*Term* (35) *of* R.

We replace

$$a \text{ by } a\left\{1 + 7\frac{\beta_2}{a^2}\gamma e\cos(\psi + h + 2g + 3l)\right\},$$

$$e \text{ by } e + \frac{7}{6}\frac{\beta_2}{a^2}\gamma\cos(\psi + h + 2g + 3l),$$

$$\gamma \text{ by } \gamma + \frac{7}{24}\frac{\beta_2}{a^2}e\cos(\psi + h + 2g + 3l),$$

$$h+g+l \text{ by } h+g+l + \frac{14}{3}\frac{\beta_2}{a^2}\gamma e\sin(\psi + h + 2g + 3l),$$

$$l \text{ by } l - \frac{7}{6}\frac{\beta_2}{a^2}\gamma\frac{1}{e}\sin(\psi + h + 2g + 3l),$$

$$h \text{ by } h + \frac{7}{24}\frac{\beta_2}{a^2}e\frac{1}{\gamma}\sin(\psi + h + 2g + 3l).$$

Operation 29.—*Term* (36) *of* R.

We replace

$$e \text{ by } e + \frac{17}{4}\frac{\beta_2}{a^2}\gamma e\cos(\psi + h + 2g + 4l),$$

$$\gamma \text{ by } \gamma + \frac{17}{32}\frac{\beta_2}{a^2}e^2\cos(\psi + h + 2g + 4l),$$

$$l \text{ by } l - \frac{17}{4}\frac{\beta_2}{a^2}\gamma\sin(\psi + h + 2g + 4l),$$

$$h \text{ by } h + \frac{17}{32}\frac{\beta_2}{a^2}e^2\frac{1}{\gamma}\sin(\psi + h + 2g + 4l),$$

a and $h+g+l$ do not change.

Operation 30.—*Term* (37) *of* R.

We replace

$$a \text{ by } a\left\{1 - \frac{\beta_2}{a^2}\gamma e\cos(\psi + h + 2g + l)\right\}.$$

$$e \text{ by } e + \frac{1}{2}\frac{\beta_2}{a^2}\gamma\cos(\psi + h + 2g + l),$$

$$\gamma \text{ by } \gamma - \frac{1}{8}\frac{\beta_2}{a^2}e\cos(\psi + h + 2g + l),$$

$$h+g+l \text{ by } h+g+l - 2\frac{\beta_2}{a^2}\gamma e\sin(\psi + h + 2g + l),$$

$$l \text{ by } l + \frac{1}{2}\frac{\beta_2}{a^2}\gamma\frac{1}{e}\sin(\psi + h + 2g + l),$$

$$h \text{ by } h - \frac{1}{8}\frac{\beta_2}{a^2}e\frac{1}{\gamma}\sin(\psi + h + 2g + l).$$

Operation 31.—*Term* (38) *of* R.

We replace

$$e \text{ by } e + \frac{\beta_2}{a^2m^2}\left[\frac{5}{9}\gamma e + \frac{23}{108}\gamma e^3 - \frac{5}{6}\gamma ee'^2 - \frac{95}{18}\gamma em + \frac{3989}{216}\gamma em^2\right]\cos(\psi + h + 2g),$$

$$\gamma \text{ by } \gamma - \frac{\beta_2}{a^2m^2}\left[\frac{5}{72}e^2 - \frac{5}{18}\gamma^2e^2 + \frac{83}{864}e^4 - \frac{5}{48}e^2e'^2 - \frac{95}{144}e^2m + \frac{403}{216}e^2m^2\right]\cos(\psi + h + 2g),$$

$$h + g + l \text{ by } h + g + l - \frac{\beta_2}{a^2m^2}\left[\frac{35}{12}\gamma e^2 - \frac{1085}{48}\gamma e^2m\right]\sin(\psi + h + 2g),$$

$$l \text{ by } l + \frac{\beta_2}{a^2m^2}\left[\frac{5}{9}\gamma - \frac{56}{27}\gamma e^2 - \frac{5}{6}\gamma e'^2 - \frac{95}{18}\gamma m + \frac{3989}{216}\gamma m^2\right]\sin(\psi + h + 2g),$$

$$h \text{ by } h - \frac{\beta_2}{a^2m^2}\left[\frac{5}{72}e^2 + \frac{83}{864}e^4 - \frac{5}{48}e^2e'^2 - \frac{95}{144}e^2m + \frac{403}{216}e^2m^2\right]\frac{1}{\gamma}\sin(\psi + h + 2g),$$

a does not change.

Operation 32.—*Term* (39) *of* R.

We replace

$$e \text{ by } e - \frac{\beta_2}{a^2m^2}\left[\frac{25}{3}\gamma^2 e - \frac{25}{24}\gamma e^3 - \frac{75}{8}\gamma em^2\right]\cos(\psi + h),$$

$$\begin{aligned}\gamma \text{ by } \gamma - \frac{\beta_2}{a^2m^2}\Big[&\frac{1}{3} - \frac{1}{6}\gamma^2 - \frac{1}{2}e'^2 - \frac{1}{24}\gamma^4 - \frac{55}{8}\gamma^2e^2 + \frac{125}{96}e^4 + \frac{1}{4}\gamma^2e'^2 + \frac{1}{8}e'^4 \\ &+ \left(\frac{1}{8} - \frac{9}{16}\gamma^2 - \frac{23}{8}e^2 - \frac{1}{18}e'^2\right)m + \left(\frac{77}{72} - \frac{169}{288}\gamma^2 - \frac{3365}{256}e^2 + \frac{19}{8}e'^2\right)m^2 \\ &+ \frac{13715}{4608}m^3 + \frac{948793}{110592}m^4 - \frac{5}{8}\frac{a^2}{a'^2} + \frac{4}{9}\frac{1}{m^2}\frac{f}{n} + \frac{1}{3}\frac{1}{m}\frac{f}{n}\Big]\cos(\psi + h),\end{aligned}$$

$$\begin{aligned}h + g + l \text{ by } h + g + l + \frac{\beta_2}{a^2m^2}\Big[&\frac{38}{3}\gamma - 7\gamma^3 - \frac{20}{3}\gamma e^2 - 19\gamma e'^2 + \left(\frac{13}{4}\gamma - \frac{135}{8}\gamma^3 - 88\gamma e^2 - \frac{13}{9}\gamma e'^2\right)m \\ &+ \frac{13513}{288}\gamma m^2 + \frac{5825}{576}\gamma m^3 + \frac{152}{9}\gamma\frac{1}{m^2}\frac{f}{n}\Big]\sin(\psi + h),\end{aligned}$$

$$l \text{ by } l + \frac{\beta_2}{a^2m^2}\left[\frac{40}{3}\gamma + \frac{185}{6}\gamma^3 - \frac{625}{24}\gamma e^2 - 20\gamma e'^2 + \frac{53}{2}\gamma m + \frac{11555}{72}\gamma m^2\right]\sin(\psi + h),$$

$$\begin{aligned}h \text{ by } h + \frac{\beta_2}{a^2m^2}\Big[&\frac{1}{3} - \frac{1}{2}\gamma^2 - \frac{1}{2}e'^2 - \frac{5}{24}\gamma^4 - \frac{355}{24}\gamma^2e^2 + \frac{125}{96}e^4 + \frac{3}{4}\gamma^2e'^2 + \frac{1}{8}e'^4 \\ &+ \left(\frac{1}{8} - \frac{11}{16}\gamma^2 - \frac{23}{8}e^2 - \frac{1}{18}e'^2\right)m + \left(\frac{77}{72} + \frac{313}{96}\gamma^2 - \frac{3365}{256}e^2 + \frac{19}{8}e'^2\right)m^2 \\ &+ \frac{13715}{4608}m^3 + \frac{948793}{110592}m^4 - \frac{5}{8}\frac{a^2}{a'^2} + \frac{4}{9}\frac{1}{m^2}\frac{f}{n} + \frac{1}{3}\frac{1}{m}\frac{f}{n}\Big]\frac{1}{\gamma}\sin(\psi + h),\end{aligned}$$

a does not change.

Operation 33.—*Term* (40) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{\beta_2}{a^3}\left[\frac{9}{32}e' - \frac{267}{256}e'm\right]\cos(\psi + h - l'),$$

$$h + g + l \text{ by } h + g + l + \frac{63}{16}\frac{\beta_2}{a^3}\gamma e' \sin(\psi + h - l'),$$

$$h \text{ by } h + \frac{\beta_2}{a^3}\left[\frac{9}{32}e' - \frac{267}{256}e'm\right]\frac{1}{\gamma}\sin(\psi + h - l'),$$

a, e, and l do not change.

Operation 34.—*Term* (41) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{27}{256}\frac{\beta_2}{a^3}e'^2\cos(\psi + h - 2l'),$$

$$h \text{ by } h + \frac{27}{256}\frac{\beta_2}{a^3}e'^2\frac{1}{\gamma}\sin(\psi + h - 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 35.—*Term* (42) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{\beta_2}{a^3}\left[\frac{9}{32}e' + \frac{189}{256}e'm\right]\cos(\psi + h + l'),$$

$$h + g + l \text{ by } h + g + l + \frac{63}{16}\frac{\beta_2}{a^3}\gamma e' \sin(\psi + h + l'),$$

$$h \text{ by } h + \frac{\beta_2}{a^3}\left[\frac{9}{32}e' + \frac{189}{256}e'm\right]\frac{1}{\gamma}\sin(\psi + h + l'),$$

a, e, and l do not change.

Operation 36.—*Term* (43) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{27}{256}\frac{\beta_2}{a^3}e'^2\cos(\psi + h + 2l'),$$

$$h \text{ by } h + \frac{27}{256}\frac{\beta_2}{a^3}\frac{e'^2}{\gamma}\sin(\psi + h + 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 37.—*Term* (44) *of* R.

We replace

$$a \text{ by } a\left\{1 - 3\frac{\beta_2}{a^3}\gamma e\cos(\psi + h + l)\right\},$$

$$e \text{ by } e - \frac{3}{2}\frac{\beta_2}{a^3}\gamma\cos(\psi + h + l),$$

$$\gamma \text{ by } \gamma + \frac{3}{8}\frac{\beta_2}{a^3}e\cos(\psi + h + l),$$

$$h + g + l \text{ by } h + g + l - 6\frac{\beta_2}{a^3}\gamma e\sin(\psi + h + l),$$

$$l \text{ by } l + \frac{3}{2}\frac{\beta_2}{a^3}\gamma\frac{1}{e}\sin(\psi + h + l),$$

$$h \text{ by } h - \frac{3}{8}\frac{\beta_2}{a^3}e\frac{1}{\gamma}\sin(\psi + h + l).$$

Operation 38.—*Term* (45) *of* R.

We replace

$$e \text{ by } e - \frac{13}{8}\frac{\beta_2}{a^3}\gamma e \cos(\psi + h + 2l),$$

$$\gamma \text{ by } \gamma + \frac{13}{64}\frac{\beta_2}{a^3}e^2 \cos(\psi + h + 2l),$$

$$l \text{ by } l + \frac{13}{8}\frac{\beta_2}{a^3}\gamma \sin(\psi + h + 2l),$$

$$h \text{ by } h - \frac{13}{64}\frac{\beta_2}{a^3}\frac{e^2}{\gamma} \sin(\psi + h + 2l),$$

a and $h+g+l$ do not change.

Operation 39.—*Term* (46) *of* R.

We replace

$$a \text{ by } a\left\{1 - 3\frac{\beta_2}{a^3}\gamma e \cos(\psi + h - l)\right\},$$

$$e \text{ by } e - \frac{3}{2}\frac{\beta_2}{a^3}\gamma \cos(\psi + h - l),$$

$$\gamma \text{ by } \gamma - \frac{3}{8}\frac{\beta_2}{a^3}e \cos(\psi + h - l),$$

$$h + g + l \text{ by } h + g + l + 6\frac{\beta_2}{a^3}\gamma e \sin(\psi + h - l),$$

$$l \text{ by } l - \frac{3}{2}\frac{\beta_2}{a^3}\frac{\gamma}{e} \sin(\psi + h - l),$$

$$h \text{ by } h + \frac{3}{8}\frac{\beta_2}{a^3}\frac{e}{\gamma} \sin(\psi + h - l).$$

Operation 40.—*Term* (47) *of* R.

We replace

$$e \text{ by } e - \frac{9}{4}\frac{\beta_2}{a^3}\gamma e \cos(\psi + h - 2l),$$

$$\gamma \text{ by } \gamma - \frac{9}{32}\frac{\beta_2}{a^3}e^2 \cos(\psi + h - 2l),$$

$$l \text{ by } l - \frac{9}{4}\frac{\beta_2}{a^3}\gamma \sin(\psi + h - 2l),$$

$$h \text{ by } h + \frac{9}{32}\frac{\beta_2}{a^3}\frac{e^2}{\gamma} \sin(\psi + h - 2l),$$

a and $h+g+l$ do not change.

Operation 41.—*Term* (48) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{3}{8}\frac{\beta_2}{a^3}\gamma^3 \cos(\psi + h - 2g - 2l),$$

$$h \text{ by } h + \frac{3}{8}\frac{\beta_2}{a^3}\gamma \sin(\psi + h - 2g - 2l),$$

a, e, $h+g+l$, and l do not change.

Operation 42.—*Term* (49) *of* R.

We replace

$$e \text{ by } e - \frac{23}{6}\frac{\beta_2}{a^2m^2}\gamma^3 e\cos(\psi + h - 2g),$$

$$\gamma \text{ by } \gamma + \frac{23}{16}\frac{\beta_2}{a^2m^2}\gamma^2 e^2\cos(\psi + h - 2g),$$

$$l \text{ by } l + \frac{23}{6}\frac{\beta_2}{a^2m^2}\gamma^3\sin(\psi + h + 2g),$$

$$h \text{ by } h - \frac{23}{16}\frac{\beta_2}{a^2m^2}\gamma e^2\sin(\psi + h - 2g),$$

a and $h+g+l$ do not change.

Operation 43.—*Term* (50) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{23}{128}\frac{\beta_2}{a^2}m^2\cos(\psi + 3h + 4g + 4l - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{23}{128}\frac{\beta_2}{a^2}\frac{m^2}{\gamma}\sin(\psi + 3h + 4g + 4l - 2h' - 2g' - 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 44.—*Term* (51) *of* R.

We replace

$$e \text{ by } e - \frac{35}{16}\frac{\beta_2}{a^2}\gamma m\cos(\psi + 3h + 4g + 3l - 2h' - 2g' - 2l'),$$

$$\gamma \text{ by } \gamma + \frac{35}{64}\frac{\beta_2}{a^2}em\cos(\psi + 3h + 4g + 3l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{35}{16}\frac{\beta_2}{a^2}\frac{\gamma}{e}m\sin(\psi + 3h + 4g + 3l - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{35}{64}\frac{\beta_2}{a^2}\frac{e}{\gamma}m\sin(\psi + 3h + 4g + 3l - 2h' - 2g' - 2l'),$$

a and $h+g+l$ do not change.

Operation 45.—*Term* (52) *of* R.

We replace

$$a \text{ by } a\left\{1 + \frac{3}{4}\frac{\beta_2}{a^2}\gamma m\cos(\psi + 3h + 2g + 2l - 2h' - 2g' - 2l')\right\},$$

$$\gamma \text{ by } \gamma - \frac{\beta_2}{a^2}\left[\frac{3}{64}m - \frac{33}{256}m^2\right]\cos(\psi + 3h + 2g + 2l - 2h' - 2g' - 2l'),$$

$$h + g + l \text{ by } h + g + l + \frac{3}{32}\frac{\beta_2}{a^2}\gamma m\sin(\psi + 3h + 2g + 2l - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{\beta_2}{a^2}\left[\frac{3}{64}m - \frac{33}{256}m^2\right]\frac{1}{\gamma}\sin(\psi + 3h + 2g + 2l - 2h' - 2g' - 2l'),$$

e and l do not change.

Operation 46.—*Term* (53) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{7}{64}\frac{\beta_2}{a^2}e'm\cos(\psi + 3h + 2g + 2l - 2h' - 2g' - 3l'),$$

$$h \text{ by } h + \frac{7}{64}\frac{\beta_2}{a^2}e'm\frac{1}{\gamma}\sin(\psi + 3h + 2g + 2l - 2h' - 2g' - 3l'),$$

a, e, $h+g+l$, and l do not change.

Operation 47.—*Term* (54) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{3}{64}\frac{\beta_2}{a^2}e'm\cos(\psi + 3h + 2g + 2l - 2h' - 2g' - l'),$$

$$h \text{ by } h - \frac{3}{64}\frac{\beta_2}{a^2}e'm\frac{1}{\gamma}\sin(\psi + 3h + 2g + 2l - 2h' - 2g' - l'),$$

a, e, $h+g+l$, and l do not change.

Operation 48.—*Term* (55) *of* R.

We replace

$$e \text{ by } e + \frac{7}{16}\frac{\beta_2}{a^2}\gamma m\cos(\psi + 3h + 2g + 3l - 2h' - 2g' - 2l'),$$

$$\gamma \text{ by } \gamma - \frac{7}{64}\frac{\beta_2}{a^2}em\cos(\psi + 3h + 2g + 3l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{7}{16}\frac{\beta_2}{a^2}\frac{\gamma}{e}m\sin(\psi + 3h + 2g + 3l - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{7}{64}\frac{\beta_2}{a^2}\frac{e}{\gamma}m\sin(\psi + 3h + 2g + 3l - 2h' - 2g' - 2l'),$$

a and $h+g+l$ do not change.

Operation 49.—*Term* (56) *of* R.

We replace

$$e \text{ by } e + 3\frac{\beta_2}{a^2}\gamma m\cos(\psi + 3h + 2g + l - 2h' - 2g' - 2l'),$$

$$\gamma \text{ by } \gamma + \frac{3}{4}\frac{\beta_2}{a^2}em\cos(\psi + 3h + 2g + l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l + 3\frac{\beta_2}{a^2}\frac{\gamma}{e}m\sin(\psi + 3h + 2g + l - 2h' - 2g' - 2l'),$$

$$h \text{ by } h - \frac{3}{4}\frac{\beta_2}{a^2}\frac{e}{\gamma}m\sin(\psi + 3h + 2g + l - 2h' - 2g' - 2l'),$$

a and $h+g+l$ do not change.

Operation 50.—*Term* (57) *of* R.

We replace

$$e \text{ by } e - \frac{15}{4}\frac{\beta_2}{a^2}\gamma e\cos(\psi + 3h + 2g - 2h' - 2g' - 2l'),$$

$$\gamma \text{ by } \gamma - \frac{15}{32}\frac{\beta_2}{a^2}e^2\cos(\psi + 3h + 2g - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{15}{4}\frac{\beta_2}{a^2}\gamma\sin(\psi + 3h + 2g - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{15}{32}\frac{\beta_2}{a^2}\frac{e^2}{\gamma}\sin(\psi + 3h + 2g - 2h' - 2g' - 2l'),$$

a and $h+g+l$ do not change.

Operation 51.—*Term* (58) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{9}{32}\frac{\beta_2}{a^3}\gamma^2 \cos(\psi + 3h - 2h' - 2g' - 2l'),$$

$$h \text{ by } h - \frac{9}{32}\frac{\beta_2}{a^3}\gamma \sin(\psi + 3h - 2h' - 2g' - 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 52.—*Term* (59) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{\beta_2}{a^3}\left[\frac{3}{64} - \frac{39}{128}\gamma^2 + \frac{3}{16}e^2 - \frac{15}{128}e'^2 - \frac{11}{512}m - \frac{127}{6144}m^2\right] \times \cos(\psi - h + 2h' + 2g' + 2l'),$$

$$h+g+l \text{ by } h+g+l - \frac{\beta_2}{a^3}\left[\frac{21}{32}\gamma - \frac{11}{256}\gamma m\right]\sin(\psi - h + 2h' + 2g' + 2l'),$$

$$l \text{ by } l + \frac{3}{4}\frac{\beta_2}{a^3}\gamma \sin(\psi - h + 2h' + 2g' + 2l'),$$

$$h \text{ by } h - \frac{\beta_2}{a^3}\left[\frac{3}{64} - \frac{117}{128}\gamma^2 + \frac{3}{16}e^2 - \frac{15}{128}e'^2 - \frac{11}{512}m - \frac{127}{6144}m^2\right]\frac{1}{\gamma} \times \sin(\psi - h + 2h' + 2g' + 2l'),$$

a and e do not change.

Operation 53.—*Term* (60) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{77}{256}\frac{\beta_2}{a^3}e'm \cos(\psi - h + 2h' + 2g' + l'),$$

$$h \text{ by } h + \frac{77}{256}\frac{\beta_2}{a^3}e'm\frac{1}{\gamma}\sin(\psi - h + 2h' + 2g' + l'),$$

a, e, $h+g+l$, and l do not change.

Operation 54.—*Term* (61) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{\beta_2}{a^2 m}\left[\frac{3}{32}e'^2 + \frac{5}{32}e'^2 m\right]\cos(\psi - h + 2h' + 2g'),$$

$$h+g+l \text{ by } h+g+l - \frac{39}{16}\frac{\beta_2}{a^2 m}\gamma e'^2 \sin(\psi - h + 2h' + 2g'),$$

$$h \text{ by } h + \frac{\beta_2}{a^2 m}\left[\frac{3}{32}e'^2 + \frac{5}{32}e'^2 m\right]\frac{1}{\gamma}\sin(\psi - h + 2h' + 2g'),$$

a, e, and l do not change.

Operation 55.—*Term* (62) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{\beta_2}{a^3}\left[\frac{7}{96}e' + \frac{23}{768}e'm\right]\cos(\psi - h + 2h' + 2g' + 3l'),$$

$$h+g+l \text{ by } h+g+l - \frac{49}{48}\frac{\beta_2}{a^3}\gamma e' \sin(\psi - h + 2h' + 2g' + 3l'),$$

$$h \text{ by } h - \frac{\beta_2}{a^3}\left[\frac{7}{96}e' + \frac{23}{768}e'm\right]\frac{1}{\gamma}\sin(\psi - h + 2h' + 2g' + 3l'),$$

a, e, and l do not change.

Operation 56.—*Term* (63) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{51}{512}\frac{\beta_2}{a^2}e'^2 \cos(\psi - h + 2h' + 2g' + 4l'),$$

$$h \text{ by } h - \frac{51}{512}\frac{\beta_2}{a^2}\frac{e'^2}{\gamma} \sin(\psi - h + 2h' + 2g' + 4l'),$$

a, e, $h+g+l$, and l do not change.

Operation 57.—*Term* (64) *of* R.

We replace

$$e \text{ by } e - \frac{3}{2}\frac{\beta_2}{a^2}\gamma m \cos(\psi - h + l + 2h' + 2g' + 2l'),$$

$$\gamma \text{ by } \gamma - \frac{3}{8}\frac{\beta_2}{a^2}em \cos(\psi - h + l + 2h' + 2g' + 2l'),$$

$$l \text{ by } l + \frac{3}{2}\frac{\beta_2}{a^2}\frac{\gamma}{e}m \sin(\psi - h + l + 2h' + 2g' + 2l'),$$

$$h \text{ by } h - \frac{3}{8}\frac{\beta_2}{a^2}\frac{e}{\gamma}m \sin(\psi - h + l + 2h' + 2g' + 2l'),$$

a and $h+g+l$ do not change.

Operation 58.—*Term* (65) *of* R.

We replace

$$e \text{ by } e - \frac{9}{16}\frac{\beta_2}{a^2}\gamma m \cos(\psi - h - l + 2h' + 2g' + 2l'),$$

$$\gamma \text{ by } \gamma + \frac{9}{64}\frac{\beta_2}{a^2}em \cos(\psi - h - l + 2h' + 2g' + 2l'),$$

$$l \text{ by } l - \frac{9}{16}\frac{\beta_2}{a^2}\frac{\gamma}{e}m \sin(\psi - h - l + 2h' + 2g' + 2l'),$$

$$h \text{ by } h + \frac{9}{64}\frac{\beta_2}{a^2}\frac{e}{\gamma}m \sin(\psi - h - l + 2h' + 2g' + 2l'),$$

a and $h+g+l$ do not change.

Operation 59.—*Term* (66) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{3}{16}\frac{\beta_2}{a^2}m^2 \cos(\psi - h - 2g - 2l + 2h' + 2g' + 2l'),$$

$$h \text{ by } h + \frac{3}{16}\frac{\beta_2}{a^2}\frac{m^2}{\gamma} \sin(\psi - h - 2g - 2l + 2h' + 2g' + 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 60.—*Term* (67) *of* R.

We replace

$$e \text{ by } e + \frac{45}{16}\frac{\beta_2}{a^2}\gamma m \cos(\psi - h - 2g - l + 2h' + 2g' + 2l'),$$

$$\gamma \text{ by } \gamma - \frac{45}{64}\frac{\beta_2}{a^2}em \cos(\psi - h - 2g - l + 2h' + 2g' + 2l'),$$

$$l \text{ by } l - \frac{45}{16}\frac{\beta_2}{a^2}\frac{\gamma}{e}m \sin(\psi - h - 2g - l + 2h' + 2g' + 2l'),$$

$$h \text{ by } h + \frac{45}{64}\frac{\beta_2}{a^2}\frac{e}{\gamma}m \sin(\psi - h - 2g - l + 2h' + 2g' + 2l'),$$

a and $h+g+l$ do not change.

Operation 61.—*Term* (68) *of* R.

We replace

$$e \text{ by } e - \frac{195}{64}\frac{\beta_2}{a^3}\gamma e \cos(\psi - h - 2g + 2h' + 2g' + 2l'),$$

$$\gamma \text{ by } \gamma + \frac{195}{512}\frac{\beta_2}{a^3}e^2 \cos(\psi - h - 2g + 2h' + 2g' + 2l'),$$

$$l \text{ by } l + \frac{195}{64}\frac{\beta_2}{a^3}\gamma \sin(\psi - h - 2g + 2h' + 2g' + 2l'),$$

$$h \text{ by } h - \frac{195}{512}\frac{\beta_2}{a^3}\frac{e^2}{\gamma} \sin(\psi - h - 2g + 2h' + 2g' + 2l'),$$

a and $h+g+l$ do not change.

Operation 62.—*Term* (69) *of* R.

We replace

$$e \text{ by } e + \frac{50}{117}\frac{\beta_2}{a^2m^3}\gamma e'\frac{a}{a'}\left[1 + \frac{8}{13}\frac{\gamma^2}{m} + \frac{10}{39}\frac{e^2}{m} - \frac{1771}{390}m\right]\cos(\psi + 2h + g - h' - g'),$$

$$\gamma \text{ by } \gamma + \frac{25}{234}\frac{\beta_2}{a^2m^3}ee'\frac{a}{a'}\left[1 + \frac{8}{13}\frac{\gamma^2}{m} + \frac{10}{39}\frac{e^2}{m} - \frac{1771}{390}m\right]\cos(\psi + 2h + g - h' - g'),$$

$$h + g + l \text{ by } h + g + l - \frac{550}{117}\frac{\beta_2}{a^2m^3}\gamma ee'\frac{a}{a'}\sin(\psi + 2h + g - h' - g'),$$

$$l \text{ by } l + \frac{50}{117}\frac{\beta_2}{a^2m^3}\gamma\frac{e'}{e}\frac{a}{a'}\left[1 + \frac{8}{13}\frac{\gamma^2}{m} + \frac{10}{13}\frac{e^2}{m} - \frac{1771}{390}m\right]\sin(\psi + 2h + g - h' - g'),$$

$$h \text{ by } h - \frac{25}{234}\frac{\beta_2}{a^2m^3}\frac{ee'}{\gamma}\frac{a}{a'}\left[1 + \frac{24}{13}\frac{\gamma^2}{m} + \frac{10}{39}\frac{e^2}{m} - \frac{1771}{390}m\right]\sin(\psi + 2h + g - h' - g'),$$

a does not change.

Operation 63.—*Term* (70) *of* R.

We replace

$$e \text{ by } e + \frac{5}{9}\frac{\beta_2}{a^2m^3}\gamma e'\frac{a}{a'}\cos(\psi + g + h' + g'),$$

$$\gamma \text{ by } \gamma - \frac{5}{36}\frac{\beta_2}{a^2m^3}ee'\frac{a}{a'}\cos(\psi + g + h' + g'),$$

$$l \text{ by } l + \frac{5}{9}\frac{\beta_2}{a^2m^3}\frac{\gamma e'}{e}\frac{a}{a'}\sin(\psi + g + h' + g'),$$

$$h \text{ by } h - \frac{5}{36}\frac{\beta_2}{a^2m^3}\frac{ee'}{\gamma}\frac{a}{a'}\sin(\psi + g + h' + g'),$$

a and $h+g+l$ do not change.

Operation 64.—*Term* (71) *of* R.

We replace

$$e \text{ by } e + \frac{5}{6}\frac{\beta_2}{a^2m^3}\gamma e'\frac{a}{a'}\cos(\psi - g + h' + g'),$$

$$\gamma \text{ by } \gamma - \frac{5}{24}\frac{\beta_2}{a^2m^3}ee'\frac{a}{a'}\cos(\psi - g + h' + g'),$$

$$l \text{ by } l - \frac{5}{6}\frac{\beta_2}{a^2m^3}\frac{\gamma e'}{e}\frac{a}{a'}\sin(\psi - g + h' + g'),$$

$$h \text{ by } h + \frac{5}{24}\frac{\beta_2}{a^2m^3}\frac{ee'}{\gamma}\frac{a}{a'}\sin(\psi - g + h' + g'),$$

a and $h+g+l$ do not change.

Operation 65.—*Term* (72) *of* R.

We replace

$$a \text{ by } a\left\{1+\frac{\beta_3}{a^3}\left(1-2\gamma^2-\frac{5}{2}e^2+\frac{5}{6}m^2\right)\cos(2\psi+2h+2g+2l)\right\},$$

$$e \text{ by } e-\frac{1}{4}\frac{\beta_3}{a^3}e\cos(2\psi+2h+2g+2l),$$

$$\gamma \text{ by } \gamma-\frac{1}{4}\frac{\beta_3}{a^3}\gamma\cos(2\psi+2h+2g+2l),$$

$$h+g+l \text{ by } h+g+l+\frac{\beta_3}{a^3}\left[\frac{3}{4}-2\gamma^2-\frac{5}{2}e^2+\frac{17}{8}m^2\right]\sin(2\psi+2h+2g+2l),$$

$$l \text{ by } l+2\frac{\beta_3}{a^3}\sin(2\psi+2h+2g+2l),$$

$$h \text{ by } h-\frac{1}{4}\frac{\beta_3}{a^3}\sin(2\psi+2h+2g+2l).$$

Operation 66.—*Term* (73) *of* R.

We replace

$$a \text{ by } a\left\{1+3\frac{\beta_3}{a^3}e'm\cos(2\psi+2h+2g+2l-l')\right\},$$

e, γ, $h+g+l$, l, and h do not change.

Operation 67.—*Term* (75) *of* R.

We replace

$$a \text{ by } a\left\{1-3\frac{\beta_3}{a^3}e'm\cos(2\psi+2h+2g+2l+l')\right\},$$

e, γ, $h+g+l$, l, and h do not change.

Operation 68.—*Term* (77) *of* R.

We replace

$$a \text{ by } a\left\{1+\frac{7}{2}\frac{\beta_3}{a^3}e\cos(2\psi+2h+2g+3l)\right\},$$

$$e \text{ by } e+\frac{\beta_3}{a^3}\left[\frac{7}{12}-\frac{7}{6}\gamma^2-\frac{235}{96}e^2+\frac{8773}{2304}m^2\right]\cos(2\psi+2h+2g+3l),$$

$$\gamma \text{ by } \gamma-\frac{7}{12}\frac{\beta_3}{a^3}\gamma e\cos(2\psi+2h+2g+3l),$$

$$h+g+l \text{ by } h+g+l+\frac{49}{24}\frac{\beta_3}{a^3}e\sin(2\psi+2h+2g+3l),$$

$$l \text{ by } l-\frac{\beta_3}{a^3}\left[\frac{7}{12}-\frac{7}{6}\gamma^2-\frac{593}{96}e^2+\frac{8773}{2304}m^2\right]\frac{1}{e}\sin(2\psi+2h+2g+3l),$$

$$h \text{ by } h-\frac{7}{12}\frac{\beta_3}{a^3}e\sin(2\psi+2h+2g+3l).$$

Operation 69.—*Term* (78) *of* R.

We replace

$$e \text{ by } e+\frac{105}{32}\frac{\beta_3}{a^3}e'm\cos(2\psi+2h+2g+3l-l'),$$

$$l \text{ by } l-\frac{105}{32}\frac{\beta_3}{a^3}\frac{e'}{e}m\sin(2\psi+2h+2g+3l-l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 70.—*Term* (79) *of* R.

We replace

$$e \text{ by } e - \frac{105}{32}\frac{\beta_3}{a^3}e'm \cos(2\psi + 2h + 2g + 3l + l'),$$

$$l \text{ by } l + \frac{105}{32}\frac{\beta_3}{a^3}\frac{e'}{e} m \sin(2\psi + 2h + 2g + 3l + l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 71.—*Term* (80) *of* R.

We replace

$$a \text{ by } a\left\{1 + \frac{17}{2}\frac{\beta_3}{a^3}e^2 \cos(2\psi + 2h + 2g + 4l)\right\},$$

$$e \text{ by } e + \frac{17}{8}\frac{\beta_3}{a^3}e \cos(2\psi + 2h + 2g + 4l),$$

$$h + g + l \text{ by } h + g + l + \frac{17}{4}\frac{\beta_3}{a^3}e^2 \sin(2\psi + 2h + 2g + 4l),$$

$$l \text{ by } l - \frac{17}{8}\frac{\beta_3}{a^3}\sin(2\psi + 2h + 2g + 4l),$$

γ and h do not change.

Operation 72.—*Term* (81) *of* R.

We replace

$$e \text{ by } e + \frac{169}{32}\frac{\beta_3}{a^3}e^2 \cos(2\psi + 2h + 2g + 5l),$$

$$l \text{ by } l - \frac{169}{32}\frac{\beta_3}{a^3}e \sin(2\psi + 2h + 2g + 5l),$$

a, γ, $h+g+l$, and h do not change.

Operation 73.—*Term* (82) *of* R.

We replace

$$a \text{ by } a\left\{1 - \frac{1}{2}\frac{\beta_3}{a^3}e \cos(2\psi + 2h + 2g + l)\right\},$$

$$e \text{ by } e + \frac{\beta_3}{a^3}\left[\frac{1}{4} - \frac{1}{2}\gamma^2 - \frac{1}{32}e^2 + \frac{1555}{768}m^2\right]\cos(2\psi + 2h + 2g + l),$$

$$\gamma \text{ by } \gamma + \frac{1}{4}\frac{\beta_3}{a^3}\gamma e \cos(2\psi + 2h + 2g + l),$$

$$h + g + l \text{ by } h + g + l - \frac{7}{8}\frac{\beta_3}{a^3}e \sin(2\psi + 2h + 2g + l),$$

$$l \text{ by } l + \frac{\beta_3}{a^3}\left[\frac{1}{4} - \frac{1}{2}\gamma^2 - \frac{35}{32}e^2 + \frac{1555}{768}m^2\right]\frac{1}{e}\sin(2\psi + 2h + 2g + l),$$

$$h \text{ by } h + \frac{1}{4}\frac{\beta_3}{a^3}e \sin(2\psi + 2h + 2g + l).$$

Operation 74.—*Term* (83) *of* R.

We replace

$$e \text{ by } e + \frac{3}{32}\frac{\beta_3}{a^3}e'm\cos(2\psi + 2h + 2g + l - l'),$$

$$l \text{ by } l + \frac{3}{32}\frac{\beta_3}{a^3}\frac{e'}{e}m\sin(2\psi + 2h + 2g + l - l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 75.—*Term* (84) *of* R.

We replace

$$e \text{ by } e - \frac{3}{32}\frac{\beta_3}{a^3}e'm\cos(2\psi + 2h + 2g + l + l'),$$

$$l \text{ by } l - \frac{3}{32}\frac{\beta_3}{a^3}\frac{e'}{e}m\sin(2\psi + 2h + 2g + l + l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 76.—*Term* (85) *of* R.

We replace

$$e \text{ by } e - \frac{\beta_3}{a^3m^3}\left[\frac{5}{3}\gamma^2e - \frac{85}{4}\gamma^2em + \frac{1}{24}em^2 - \frac{625}{64}em^3\right]\cos(2\psi + 2h + 2g),$$

$$h+g+l \text{ by } h+g+l + \frac{\beta_3}{a^3m^3}\left[\frac{55}{6}\gamma^2e^2 + \frac{1}{12}e^2m^2\right]\sin(2\psi + 2h + 2g),$$

$$l \text{ by } l - \frac{\beta_3}{a^3m^3}\left[\frac{5}{3}\gamma^2 - \frac{85}{4}\gamma^2m + \frac{1}{24}m^2 - \frac{625}{64}m^3\right]\sin(2\psi + 2h + 2g),$$

$$h \text{ by } h + \frac{\beta_3}{a^3m^3}\left[\frac{5}{12}e^2 - \frac{85}{16}e^2m\right]\sin(2\psi + 2h + 2g),$$

a and γ do not change.

Operation 77.—*Term* (88) *of* R.

We replace

$$e \text{ by } e + \frac{1}{32}\frac{\beta_3}{a^3}e^2\cos(2\psi + 2h + 2g - l),$$

$$l \text{ by } l + \frac{1}{32}\frac{\beta_3}{a^3}e\sin(2\psi + 2h + 2g - l),$$

a, γ, $h+g+l$, and h do not change.

Operation 78.—*Term* (89) *of* R.

We replace

$$e \text{ by } e + \frac{35}{24}\frac{\beta_3}{a^3}\gamma^2\cos(2\psi + 2h + 4g + 3l),$$

$$\gamma \text{ by } \gamma - \frac{35}{48}\frac{\beta_3}{a^3}\gamma e\cos(2\psi + 2h + 4g + 3l),$$

$$l \text{ by } l + \frac{35}{24}\frac{\beta_3}{a^3}\frac{\gamma^2}{e}\sin(2\psi + 2h + 4g + 3l),$$

$$h \text{ by } h - \frac{35}{48}\frac{\beta_3}{a^3}e\sin(2\psi + 2h + 4g + 3l),$$

a and $h+g+l$ do not change.

Operation 79.—*Term* (90) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{\beta_3}{a^2m^3}\Big[\frac{1}{3}\gamma + \frac{1}{3}\gamma^3 - \frac{1}{2}\gamma e'^2 + \Big(\frac{1}{8}\gamma - \frac{3}{8}\gamma^3 - \frac{23}{8}\gamma e^2 - \frac{1}{18}\gamma e'^2\Big)m$$
$$+ \frac{10}{9}\gamma m^2 + \frac{13319}{4608}\gamma m^3 + \frac{4}{9}\frac{\gamma}{m^2}\frac{f}{n}\Big] \cos(2\psi + 2h),$$

$$h + g + l \text{ by } h + g + l - \frac{\beta_3}{a^2m^3}\Big[\frac{20}{3}\gamma^2 + \frac{22}{3}\gamma^4 - \frac{10}{3}\gamma^2 e^2 - 10\gamma^2 e'^2 + \frac{7}{4}\gamma^2 m$$
$$+ \frac{3785}{144}\gamma^2 m^2\Big] \sin(2\psi + 2h),$$

$$l \text{ by } l - \frac{\beta_3}{a^2m^3}\Big[\frac{20}{3}\gamma^2 + \frac{53}{4}\gamma^2 m\Big] \sin 2\psi + 2h),$$

$$h \text{ by } h - \frac{\beta_3}{a^2m^3}\Big[\frac{1}{3} + \frac{2}{3}\gamma^2 - \frac{1}{2}e'^2 + \Big(\frac{1}{8} - \frac{3}{4}\gamma^2 - \frac{23}{8}e^2 - \frac{1}{18}e'^2\Big)m$$
$$+ \frac{10}{9}m^2 + \frac{13319}{4608}m^3 + \frac{4}{9}\frac{1}{m^2}\frac{f}{n}\Big] \sin(2\psi + 2h),$$

a and e do not change.

Operation 80.—*Term* (91) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{9}{8}\frac{\beta_3}{a^3}\gamma e' \cos(2\psi + 2h - l'),$$

$$h \text{ by } h - \frac{9}{8}\frac{\beta_3}{a^3}e' \sin(2\psi + 2h - l'),$$

a, e, $h + g + l$, and l do not change.

Operation 81.—*Term* (92) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{9}{8}\frac{\beta_3}{a^3}\gamma e' \cos(2\psi + 2h + l'),$$

$$h \text{ by } h - \frac{9}{8}\frac{\beta_3}{a^3}e' \sin(2\psi + 2h + l'),$$

a, e, $h + g + l$, and l do not change.

Operation 82.—*Term* (93) *of* R.

We replace

$$e \text{ by } e + \frac{17}{8}\frac{\beta_3}{a^3}\gamma^2 \cos(2\psi + 2h + l),$$

$$\gamma \text{ by } \gamma - \frac{17}{16}\frac{\beta_3}{a^3}\gamma e \cos(2\psi + 2h + l),$$

$$l \text{ by } l - \frac{17}{8}\frac{\beta_3}{a^3}\frac{\gamma^2}{e} \sin(2\psi + 2h + l),$$

$$h \text{ by } h + \frac{17}{16}\frac{\beta_3}{a^3}e \sin(2\psi + 2h + l),$$

a and $h + g + l$ do not change.

Operation 83.—*Term* (94) *of* R.

We replace

$$e \text{ by } e + \frac{3}{2}\frac{\beta_3}{a^3}\gamma^2 \cos(2\psi + 2h - l),$$

$$\gamma \text{ by } \gamma + \frac{3}{4}\frac{\beta_3}{a^3}\gamma e \cos(2\psi + 2h - l),$$

$$l \text{ by } l + \frac{3}{2}\frac{\beta_3}{a^3}\frac{\gamma^2}{e} \sin(2\psi + 2h - l),$$

$$h \text{ by } h - \frac{3}{4}\frac{\beta_3}{a^3} e \sin(2\psi + 2h - l),$$

a and $h + g + l$ do not change.

Operation 84.—*Term* (95) *of* R.

We replace

$$a \text{ by } a\left\{1 + \frac{23}{8}\frac{\beta_3}{a^3}m^2 \cos(2\psi + 4h + 4g + 4l - 2h' - 2g' - 2l')\right\},$$

$$h + g + l \text{ by } h + g + l - \frac{69}{64}\frac{\beta_3}{a^3}m^2 \sin(2\psi + 4h + 4g + 4l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{255}{32}\frac{\beta_3}{a^3}m \sin(2\psi + 4h + 4g + 4l - 2h' - 2g' - 2l'),$$

e, γ, and h do not change.

Operation 85.—*Term* (98) *of* R.

We replace

$$e \text{ by } e + \frac{93}{64}\frac{\beta_3}{a^3}m^2 \cos(2\psi + 4h + 4g + 5l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{93}{64}\frac{\beta_3}{a^3}\frac{m^2}{e} \sin(2\psi + 4h + 4g + 5l - 2h' - 2g' - 2l'),$$

a, γ, $h + g + l$, and h do not change.

Operation 86.—*Term* (99) *of* R.

We replace

$$a \text{ by } a\left\{1 + \frac{105}{16}\frac{\beta_3}{a^3}em \cos(2\psi + 4h + 4g + 3l - 2h' - 2g' - 2l')\right\},$$

$$e \text{ by } e - \frac{\beta_3}{a^3}\left[\frac{35}{32}m + \frac{1753}{384}m^2\right] \cos(2\psi + 4h + 4g + 3l - 2h' - 2g' - 2l'),$$

$$h + g + l \text{ by } h + g + l + \frac{35}{64}\frac{\beta_3}{a^3}em \sin(2\psi + 4h + 4g + 3l - 2h' - 2g' - 2l'),$$

$$l \text{ by } l - \frac{\beta_3}{a^3}\left[\frac{35}{32}m + \frac{1753}{384}m^2\right]\frac{1}{e} \sin(2\psi + 4h + 4g + 3l - 2h' - 2g' - 2l').$$

γ and h does not change.

Operation 87.—*Term* (100) *of* R.

We replace

$$e \text{ by } e - \frac{245}{96}\frac{\beta_3}{a^3}e'm \cos(2\psi + 4h + 4g + 3l - 2h' - 2g' - 3l'),$$

$$l \text{ by } l - \frac{245}{96}\frac{\beta_3}{a^3}\frac{e'm}{e} \sin(2\psi + 4h + 4g + 3l - 2h' - 2g' - 3l'),$$

a, γ, $h + g + l$, and h do not change.

Operation 88.—*Term* (101) *of* R.

We replace

$$e \text{ by } e + \frac{35}{32}\frac{\beta_3}{a^3}e'm \cos(2\psi + 4h + 4g + 3l - 2h' - 2g' - l'),$$

$$l \text{ by } l + \frac{35}{32}\frac{\beta_3}{a^3}\frac{e'm}{e} \sin(2\psi + 4h + 4g + 3l - 2h' - 2g' - l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 89.—*Term* (102) *of* R.

We replace

$$e \text{ by } e + \frac{15}{16}\frac{\beta_3}{a^3}em \cos(2\psi + 4h + 4g + 2l - 2h' - 2g' - 2l')$$

$$l \text{ by } l + \frac{15}{16}\frac{\beta_3}{a^3}m \sin(2\psi + 4h + 4g + 2l - 2h' - 2g' - 2l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 90.—*Term* (103) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{3}{16}\frac{\beta_3}{a^3}\gamma m \cos(2\psi + 4h + 2g + 2l - 2h' - 2g' - 2l'),$$

$$h \text{ by } h + \frac{3}{16}\frac{\beta_3}{a^3}m \sin(2\psi + 4h + 2g + 2l - 2h' - 2g' - 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 91.—*Term* (104) *of* R.

We replace

$$h+g+l \text{ by } h+g+l + \frac{\beta_3}{a^3}\left[\frac{3}{2}\gamma^2 + \frac{1}{8}m^2\right] \sin(2\psi + 2h' + 2g' + 2l'),$$

$$h \text{ by } h + \frac{\beta_3}{a^3}\left[\frac{3}{16} - \frac{1}{64}m\right] \sin(2\psi + 2h' + 2g' + 2l'),$$

a, e, γ, and l do not change.

Operation 92.—*Term* (107) *of* R.

We replace

$$h \text{ by } h + \frac{7}{24}\frac{\beta_3}{a^3}e' \sin(2\psi + 2h' + 2g' + 3l'),$$

a, e, γ, $h+g+l$, and l do not change.

Operation 93.—*Term* (109) *of* R.

We replace

$$a \text{ by } a\left\{1 - \frac{15}{16}\frac{\beta_3}{a^3}em \cos(2\psi + l + 2h' + 2g' + 2l')\right\},$$

$$e \text{ by } e - \frac{\beta_3}{a^3}\left[\frac{15}{32}m + \frac{391}{128}m^2\right] \cos(2\psi + l + 2h' + 2g' + 2l'),$$

$$h+g+l \text{ by } h+g+l - \frac{15}{64}\frac{\beta_3}{a^3}em \sin(2\psi + l + 2h' + 2g' + 2l'),$$

$$l \text{ by } l + \frac{\beta_3}{a^3}\left[\frac{15}{32}m + \frac{391}{128}m^2\right]\frac{1}{e} \sin(2\psi + l + 2h' + 2g' + 2l'),$$

γ and h do not change.

Operation 94.—*Term* (110) *of* R.

We replace

$$e \text{ by } e + \frac{15}{32}\frac{\beta_3}{a^3}e'm\cos(2\psi + l + 2h' + 2g' + l'),$$

$$l \text{ by } l - \frac{15}{32}\frac{\beta_3}{a^3}\frac{e'm}{e}\sin(2\psi + l + 2h' + 2g' + l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 95.—*Term* (111) *of* R.

We replace

$$e \text{ by } e - \frac{35}{32}\frac{\beta_3}{a^3}e'm\cos(2\psi + l + 2h' + 3g' + 3l'),$$

$$l \text{ by } l + \frac{35}{32}\frac{\beta_3}{a^3}\frac{e'm}{e}\sin(2\psi + l + 2h' + 3g' + 3l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 96.—*Term* (112) *of* R.

We replace

$$e \text{ by } e - \frac{15}{4}\frac{\beta_3}{a^3}em\cos(2\psi + 2l + 2h' + 2g' + 2l'),$$

$$l \text{ by } l + \frac{15}{4}\frac{\beta_3}{a^3}m\sin(2\psi + 2l + 2h' + 2g' + 2l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 97.—*Term* (113) *of* R.

We replace

$$e \text{ by } e + \frac{5}{64}\frac{\beta_3}{a^3}m^2\cos(2\psi - l + 2h' + 2g' + 2l'),$$

$$l \text{ by } l + \frac{5}{64}\frac{\beta_3}{a^3}\frac{m^2}{e}\sin(2\psi - l + 2h' + 2g' + 2l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 98.—*Term* (114) *of* R.

We replace

$$\gamma \text{ by } \gamma - \frac{3}{8}\frac{\beta_3}{a^3}\gamma m\cos(2\psi + 2g + 2l + 2h' + 2g' + 2l'),$$

$$h \text{ by } h - \frac{3}{8}\frac{\beta_3}{a^3}m\sin(2\psi + 2g + 2l + 2h' + 2g' + 2l'),$$

a, e, $h+g+l$, and l do not change.

Operation 99.—*Term* (115) *of* R.

We replace

$$e \text{ by } e - \frac{225}{128}\frac{\beta_3}{a^3}em\cos(2\psi - 2h - 2g + 4h' + 4g' + 4l'),$$

$$l \text{ by } l + \frac{225}{128}\frac{\beta_3}{a^3}m\sin(2\psi - 2h - 2g + 4h' + 4g' + 4l'),$$

a, γ, $h+g+l$, and h do not change.

Operation 100.—*Term* (116) *of* R.

We replace

$$\gamma \text{ by } \gamma + \frac{9}{512}\frac{\beta_3}{a^3}\gamma m \cos(2\psi - 2h + 4h' + 4g' + 4l'),$$

$$h \text{ by } h + \frac{9}{512}\frac{\beta_3}{a^3} m \sin(2\psi - 2h + 4h' + 4g' + 4l'),$$

a, e, $h+g+l$, and l do not change.

Operation 101.—*Term* (122) *of* R.

We replace

$$e \text{ by } e + \frac{45}{4}\frac{\beta_3}{a^2 m} e' \frac{a}{a'} \cos(2\psi + h + g + h' + g'),$$

$$l \text{ by } l + \frac{45}{4}\frac{\beta_3}{a^2 m}\frac{e'}{e}\frac{a}{a'} \sin(2\psi + h + g + h' + g'),$$

a, γ, $h+g+l$, and h do not change.

After these transformations are executed, the mean value of $\frac{d(h+g+l)}{dt}$ is no longer n, nor do the coefficients of $\sin l$ and $\sin F$, in V and U, respectively, have the same values as in the elliptic theory. In order to make them have the same values, we perform the following additional operation:

Operation 102.

We replace

$$a \text{ by } a\left\{1 + \frac{4}{3}\frac{\beta_1}{a^3}\right\},$$

$$e \text{ by } e - \frac{\beta_1}{a^3}\left[\frac{3}{2}e + \frac{225}{64}em\right],$$

$$\gamma \text{ by } \gamma + \frac{\beta_1}{a^3}\left[\frac{1}{2}\gamma + \frac{9}{64}\gamma m\right],$$

l, g, and h do not change.

The following operation was omitted at its proper place:

Operation 103.—*Term* (105) *of* R.

We replace

$$h \text{ by } h - \frac{3}{8}\frac{\beta_3}{a^3} e' \sin(2\psi + 2h' + 2g' + l'),$$

a, e, γ, $h+g+l$, and l do not change.

CHAPTER III.

DETAIL OF THE NEW TERMS WHICH ARISE IN THE CO-ORDINATES OF THE MOON THROUGH THE PRECEDING OPERATIONS.

The substitutions indicated in the preceding operations must be made in the following expressions of V, U, and $\frac{a}{r}$, taken from DELAUNAY'S second volume. The rules for selecting the terms to be retained are so simple that they need not be mentioned.

(0) $V = h + g + l$

(1) $+\left[-\left(3e' - \frac{27}{2}\gamma^2 e' + \frac{27}{8}e^2 e'\right)m - \frac{117}{8}\gamma^2 e' m^2\right]\sin l'$

(2) $-\left(\frac{9}{4}e'^2 - \frac{81}{8}\gamma^2 e'^2\right)m \sin 2l'$

(3) $+\left[2e - \frac{1}{4}e^3\right]\sin l$

(4) $+\left[\left(\frac{21}{4}ee' - \frac{63}{2}\gamma^2 ee'\right)m + \frac{1233}{32}ee'm^2\right]\sin(l - l')$

(5) $+\frac{63}{16}ee'^2 m \sin(l - 2l')$

(6) $+\left[-\left(\frac{21}{4}ee' - \frac{63}{2}\gamma^2 ee'\right)m - \frac{717}{32}ee'm^2\right]\sin(l + l')$

(7) $-\frac{63}{16}ee'^2 m \sin(l + 2l')$

(8) $+\left[\frac{5}{4}e^2 - \frac{5}{4}\gamma^2 e^2 - \frac{11}{24}e^4 + \frac{135}{32}\gamma^2 e^2 m - \frac{7}{16}e^2 m^2\right]\sin 2l$

(9) $+\frac{105}{16}e^2 e' m \sin(2l - l')$

(10) $-\frac{105}{16}e^2 e' m \sin(2l + l')$

(11) $+\left[\frac{13}{12}e^3 - \frac{5}{2}\gamma^2 e^3\right]\sin 3l$

(12) $+\frac{103}{96}e^4 \sin 4l$

(13) $+\left[-\gamma^2 - \gamma^4 - \frac{9}{4}\gamma^2 e^2 + \frac{675}{32}\gamma^2 e^2 m + \frac{11}{4}\gamma^2 m^2 - \frac{231}{64}\gamma^2 m^3\right]\sin(2g + 2l)$

(14) $+\left[-\frac{3}{4}\gamma^2 e' m + \frac{123}{32}\gamma^2 e' m^2\right]\sin(2g + 2l - l')$

(15) $-\frac{9}{16}\gamma^2 e'^2 m \sin(2g + 2l - 2l')$

(16) $+\left[\frac{3}{4}\gamma^2 e'm + \frac{201}{32}\gamma^2 e'm^2\right]\sin(2g+2l+l')$

(17) $+\frac{9}{16}\gamma^2 e'^2 m \sin(2g+2l+2l')$

(18) $+\left[-2\gamma^2 e - 2\gamma^4 e - \frac{11}{8}\gamma^2 e^3 + \frac{19}{4}\gamma^2 em^2\right]\sin(2g+3l)$

(19) $-\frac{27}{4}\gamma^2 ee'm \sin(2g+3l-l')$

(20) $+\frac{27}{4}\gamma^2 ee'm \sin(2g+3l+l')$

(21) $-\frac{13}{4}\gamma^2 e^3 \sin(2g+4l)$

(22) $-\frac{59}{12}\gamma^2 e^3 \sin(2g+5l)$

(23) $+\left[-3\gamma^2 e - 18\gamma^4 e + \frac{61}{8}\gamma^2 e^3 + \frac{135}{8}\gamma^2 em + \frac{213}{64}\gamma^2 em^2\right]\sin(2g+l)$

(24) $+\frac{45}{8}\gamma^2 ee'm \sin(2g+l-l')$

(25) $-\frac{45}{8}\gamma^2 ee'm \sin(2g+l+l')$

(26) $+\left[\frac{1}{2}\gamma^2 e^2 + \frac{135}{16}\gamma^2 e^2 m\right]\sin 2g$

(27) $+\frac{7}{6}\gamma^2 e^3 \sin(2g-l)$

(28) $+\frac{1}{2}\gamma^4 \sin(4g+4l)$

(29) $+2\gamma^4 e \sin(4g+5l)$

(30) $+3\gamma^4 e \sin(4g+3l)$

(31) $+\Big[\left(-\frac{3}{4}\gamma^2 + \frac{75}{16}e^2 - \frac{9}{4}\gamma^4 - \frac{63}{8}\gamma^2 e^2 + \frac{15}{8}\gamma^2 e'^2\right)m$

$\quad + \left(\frac{11}{8} - \frac{47}{16}\gamma^2 + \frac{1101}{64}e^2 - \frac{55}{16}e'^2\right)m^2 + \left(\frac{59}{12} - \frac{5149}{768}\gamma^2\right)m^3\Big]\sin 2D$

(32) $+\left[\left(-\frac{7}{4}\gamma^2 e' + \frac{175}{16}e^2 e'\right)m + \left(\frac{77}{16}e' - \frac{209}{16}\gamma^2 e'\right)m^2\right]\sin(2D-l')$

(33) $-\frac{51}{16}\gamma^2 e'^2 m \sin(2D-2l')$

(34) $+\left[\left(\frac{3}{4}\gamma^2 e' - \frac{75}{16}e^2 e'\right)m - \left(\frac{11}{16}e' - \frac{73}{16}\gamma^2 e'\right)m^2\right]\sin(2D+l')$

(35) $+\frac{9}{16}\gamma^2 e'^2 m \sin(2D+2l')$

(36) $+\left[\left(-\frac{3}{2}\gamma^2 e + \frac{195}{32}e^3\right)m + \left(\frac{17}{8}e - \frac{41}{8}\gamma^2 e\right)m^2 + \frac{169}{24}em^3\right]\sin(2D+l)$

(37) $+\left[-\frac{7}{2}\gamma^2 ee'm + \frac{119}{16}ee'm^2\right]\sin(2D+l-l')$

(38) $+\left[\frac{3}{2}\gamma^2 ee'm - \frac{17}{16}ee'm^2\right]\sin(2D+l+l')$

(39) $$+\left[-\frac{39}{16}\gamma^2e^2m+\frac{95}{32}e^2m^2\right]\sin(2D+2l)$$

(40) $$+\left[\left(\frac{15}{4}e-6\gamma^2e-\frac{75}{8}ee'^2\right)m+\left(\frac{263}{16}e-\frac{359}{8}\gamma^2e\right)m^2+\frac{48217}{768}em^3\right]\sin(2D-l)$$

(41) $$+\left[\left(\frac{35}{4}ee'-14\gamma^2ee'\right)m+\frac{1801}{32}ee'm^2\right]\sin(2D-l-l')$$

(42) $$+\frac{255}{16}ee'^2m\sin(2D-l-2l')$$

(43) $$+\left[-\left(\frac{15}{4}ee'-6\gamma^2ee'\right)m-\frac{173}{32}ee'm^2\right]\sin(2D-l+l')$$

(44) $$-\frac{45}{16}ee'^2m\sin(2D-l+2l')$$

(45) $$+\left[\left(\frac{45}{16}e^2-\frac{3}{2}\gamma^2e^2\right)m+\frac{53}{4}e^2m^2\right]\sin(2D-2l)$$

(46) $$+\frac{105}{16}e^2e'm\sin(2D-2l-l')$$

(47) $$-\frac{45}{16}e^2e'm\sin(2D-2l+l')$$

(48) $$+\frac{105}{32}e^3m\sin(2D-3l)$$

(49) $$+\left[\left(\frac{3}{4}\gamma^4-\frac{195}{16}\gamma^2e^2\right)m-\frac{11}{8}\gamma^2m^2-\frac{59}{12}\gamma^2m^3\right]\sin(2D+2F)$$

(50) $$-\frac{77}{16}\gamma^2e'm^2\sin(2D+2F-l')$$

(51) $$+\frac{11}{16}\gamma^2e'm^2\sin(2D+2F+l')$$

(52) $$-\frac{39}{8}\gamma^2em^2\sin(2D+2F+l)$$

(53) $$+\left[-\frac{15}{4}\gamma^2em-19\gamma^2em^2\right]\sin(2D+2F-l)$$

(54) $$-\frac{35}{4}\gamma^2ee'm\sin(2D+2F-l-l')$$

(55) $$+\frac{15}{4}\gamma^2ee'm\sin(2D+2F-l+l')$$

(56) $$-\frac{15}{2}\gamma^2e^2m\sin(2D+2F-2l)$$

(57) $$+\left[\left(\frac{9}{4}\gamma^2-\frac{3}{2}\gamma^4-\frac{75}{8}\gamma^2e^2-\frac{45}{8}\gamma^2e'^2\right)m-\frac{11}{2}\gamma^2m^2-\frac{2939}{768}\gamma^2m^3\right]\sin(2D-2F)$$

(58) $$+\left[\frac{21}{4}\gamma^2e'm-11\gamma^2e'm^2\right]\sin(2D-2F-l')$$

(59) $$+\frac{153}{16}\gamma^2e'^2m\sin(2D-2F-2l')$$

(60) $$+\left[-\frac{9}{4}\gamma^2e'm-\frac{59}{8}\gamma^2e'm^2\right]\sin(2D-2F+l')$$

(61) $$-\frac{27}{16}\gamma^2e'^2m\sin(2D-2F+2l')$$

(62) $+\left[-\frac{33}{8}\gamma^2 em+\frac{231}{64}\gamma^2 em^2\right]\sin(2D-2F+l)$

(63) $-\frac{77}{8}\gamma^2 ee'm\sin(2D-2F+l-l')$

(64) $+\frac{33}{8}\gamma^2 ee'm\sin(2D-2F+l+l')$

(65) $-\frac{45}{8}\gamma^2 e^2 m\sin(2D-2F+2l)$

(66) $+\left[\frac{3}{2}\gamma^2 em-\frac{61}{4}\gamma^2 em^2\right]\sin(2D-2F-l)$

(67) $+\frac{7}{2}\gamma^2 ee'm\sin(2D-2F-l-l')$

(68) $-\frac{3}{2}\gamma^2 ee'm\sin(2D-2F-l+l')$

(69) $-\frac{15}{8}\gamma^2 e^2 m\sin(2D-2F-2l)$

(70) $-\frac{3}{2}\gamma^4 m\sin(2D-4F)$

(71) $-\frac{33}{32}\gamma^2 m^2\sin 4D$

(72) $+\left[-\frac{45}{16}\gamma^2 em^2+\frac{255}{64}em^2\right]\sin(4D-l)$

(73) $+\frac{1125}{256}e^2 m^2\sin(4D-2l)$

(74) $+\left[-\frac{9}{64}\gamma^2 m^2+\frac{255}{128}\gamma^2 m^2\right]\sin(4D-2F)$

(75) $-\frac{21}{32}\gamma^2 e'm^2\sin(4D-2F-l')$

(76) $+\frac{9}{32}\gamma^2 e'm^2\sin(4D-2F+l')$

(77) $-\frac{9}{32}\gamma^2 em^2\sin(4D-2F+l)$

(78) $+\frac{99}{32}\gamma^2 em^2\sin(4D-2F-l)$

(79) $-\left[\frac{15}{8}-\frac{165}{8}\gamma^2\right]m\frac{a}{a'}\sin D$

(80) $+\left[\frac{5}{2}e'-\frac{15}{2}\gamma^2 e'\right]\frac{a}{a'}\sin(D+l')$

(81) $-\frac{75}{32}em\frac{a}{a'}\sin(D+l)$

(82) $+\frac{25}{8}ee'\frac{a}{a'}\sin(D+l+l')$

(83) $-\frac{165}{32}em\frac{a}{a'}\sin(D-l)$

(84) $+\frac{25}{8}ee'\frac{a}{a'}\sin(D-l+l')$

(85) $+\frac{15}{8}\gamma^2 m \frac{a}{a'} \sin(D + 2F)$

(86) $-\frac{5}{2}\gamma^2 e' \frac{a}{a'} \sin(D + 2F + l')$

(87) $-\frac{75}{8}\gamma^2 m \frac{a}{a'} \sin(D - 2F)$

(88) $+\frac{5}{6}\gamma^2 e' \frac{a}{a'} \sin(D - 2F + l')$

(89) $-\frac{25}{8}\gamma^2 m \frac{a}{a'} \sin(3D - 2F).$

(1) $U = \left[2\gamma - 2\gamma e^2 - \frac{1}{4}\gamma^3 + \frac{7}{32}\gamma e^4\right] \sin F$

(2) $+\left[\left(\frac{3}{4}\gamma e' - 9\gamma^3 e' - \frac{15}{8}\gamma e^2 e' + \frac{27}{32}\gamma e'^3\right) m + \frac{9}{32}\gamma e' m^2 - \frac{1107}{32}\gamma e' m^3\right] \sin(F - l')$

(3) $+\left[\frac{9}{16}\gamma e'^2 m - \frac{45}{128}\gamma e'^2 m^2\right] \sin(F - 2l')$

(4) $+\frac{53}{96}\gamma e'^3 m \sin(F - 3l')$

(5) $+\left[-\left(\frac{3}{4}\gamma e' - 9\gamma^3 e' - \frac{15}{8}\gamma e^2 e' + \frac{27}{32}\gamma e'^3\right) m - \frac{69}{32}\gamma e' m^2 + \frac{2369}{64}\gamma e' m^3\right] \sin(F + l')$

(6) $+\left[-\frac{9}{16}\gamma e'^2 m - \frac{309}{128}\gamma e'^2 m^2\right] \sin(F + 2l')$

(7) $-\frac{53}{96}\gamma e'^3 m \sin(F + 3l')$

(8) $+\left[2\gamma e - \frac{5}{2}\gamma e^3 - \frac{1}{2}\gamma e m^2 - \frac{21}{8}\gamma e m^3\right] \sin(F + l)$

(9) $+\left[6\gamma e e' m + \frac{609}{16}\gamma e e' m^2\right] \sin(F + l - l')$

(10) $+\frac{9}{2}\gamma e e'^2 m \sin(F + l - 2l')$

(11) $+\left[-6\gamma e e' m - \frac{405}{16}\gamma e e' m^2\right] \sin(F + l + l')$

(12) $-\frac{9}{2}\gamma e e'^2 m \sin(F + l + 2l')$

(13) $+\left[\frac{9}{4}\gamma e^2 - \frac{5}{8}\gamma^3 e^2 - \frac{27}{8}\gamma e^4 - \frac{17}{16}\gamma e^2 m^2\right] \sin(F + 2l)$

(14) $+\frac{405}{32}\gamma e^2 e' m \sin(F + 2l - l')$

(15) $-\frac{405}{32}\gamma e^2 e' m \sin(F + 2l + l')$

(16) $+\frac{8}{3}\gamma e^3 \sin(F + 3l)$

(17) $+\frac{625}{192}\gamma e^4 \sin(F + 4l)$

(18) $$+\left[-2\gamma e-5\gamma^3 e+\frac{5}{4}\gamma e^3+\left(\frac{135}{8}\gamma^3 e-\frac{135}{32}\gamma e^3\right)m+\frac{189}{32}\gamma e m^2+\frac{375}{32}\gamma e m^3\right]\sin(F-l)$$

(19) $$+\left[\frac{9}{2}\gamma e e' m+\frac{123}{4}\gamma e e' m^2\right]\sin(F-l-l')$$

(20) $$+\frac{27}{8}\gamma e e'^2 m\sin(F-l-2l')$$

(21) $$+\left[-\frac{9}{2}\gamma e e' m-\frac{111}{4}\gamma e e' m^2\right]\sin(F-l+l')$$

(22) $$-\frac{27}{8}\gamma e e'^2 m\sin(F-l+2l')$$

(23) $$+\left[-\frac{3}{2}\gamma e^2-10\gamma^3 e^2+\frac{77}{48}\gamma e^4+\frac{135}{32}\gamma e^2 m+\frac{2025}{256}\gamma e^2 m^2\right]\sin(F-2l)$$

(24) $$+\frac{117}{16}\gamma e^2 e' m\sin(F-2l-l')$$

(25) $$-\frac{117}{16}\gamma e^2 e' m\sin(F-2l+l')$$

(26) $$+\left[-\frac{17}{12}\gamma e^3+\frac{135}{32}\gamma e^3 m\right]\sin(F-3l)$$

(27) $$-\frac{99}{64}\gamma e^4\sin(F-4l)$$

(28) $$+\left[-\frac{1}{3}\gamma^3-\frac{1}{4}\gamma^5-\frac{33}{4}\gamma^3 e^2+\frac{11}{4}\gamma^3 m^2\right]\sin 3F$$

(29) $$-\frac{3}{8}\gamma^3 e' m\sin(3F-l')$$

(30) $$+\frac{3}{8}\gamma^3 e' m\sin(3F+l')$$

(31) $$-\gamma^3 e\sin(3F+l)$$

(32) $$-\frac{17}{8}\gamma^3 e^2\sin(3F+2l)$$

(33) $$+\left[-4\gamma^3 e+\frac{135}{8}\gamma^3 e m\right]\sin(3F-l)$$

(34) $$+\frac{13}{8}\gamma^3 e^2\sin(3F-2l)$$

(35) $$+\frac{3}{20}\gamma^5\sin 5F$$

(36) $$+\left[\left(-\frac{3}{8}\gamma^3+\frac{135}{16}\gamma e^2\right)m+\left(\frac{11}{8}\gamma-\frac{91}{32}\gamma^3+\frac{1929}{64}\gamma e^2-\frac{55}{16}\gamma e'^2\right)m^2+\frac{59}{12}\gamma m^3+\frac{7063}{576}\gamma m^4\right]\sin(2D+F)$$

(37) $$+\left[\left(-\frac{7}{8}\gamma^3 e'+\frac{315}{16}\gamma e^2 e'\right)m+\frac{77}{16}\gamma e' m^2+\frac{1949}{64}\gamma e' m^3\right]\sin(2D+F-l')$$

(38) $$+\frac{187}{16}\gamma e'^2 m^2\sin(2D+F-2l')$$

(39) $$+\left[\left(\frac{3}{8}\gamma^3 e'-\frac{135}{16}\gamma e^2 e'\right)m-\frac{11}{16}\gamma e' m^2-\frac{1127}{192}\gamma e' m^3\right]\sin(2D+F+l')$$

(40) $$+\left[\left(-\frac{9}{8}\gamma^3 e+15\gamma e^3\right)m+\frac{7}{2}\gamma e m^2+\frac{287}{24}\gamma e m^3\right]\sin(2D+F+l)$$

(41) $$+\frac{49}{4}\gamma e e' m^2\sin(2D+F+l-l')$$

(42) $$-\frac{7}{4}\gamma e e' m^2\sin(2D+F+l+l')$$

(43) $$+\frac{425}{64}\gamma e^2 m^2\sin(2D+F+2l)$$

(44) $$+\left[\left(\frac{15}{4}\gamma e-\frac{33}{4}\gamma^3 e-\frac{165}{32}\gamma e^3-\frac{75}{8}\gamma e e'^2\right)m+\frac{241}{16}\gamma e m^2+\frac{43721}{768}\gamma e m^3\right]$$
$$\times\sin(2D+F-l)$$

(45) $$+\left[\frac{35}{4}\gamma e e' m+\frac{423}{8}\gamma e e' m^2\right]\sin(2D+F-l-l')$$

(46) $$+\frac{255}{16}\gamma e e'^2 m\sin(2D+F-l-2l')$$

(47) $$+\left[-\frac{15}{4}\gamma e e' m-\frac{49}{8}\gamma e e' m^2\right]\sin(2D+F-l+l')$$

(48) $$-\frac{45}{16}\gamma e e'^2 m\sin(2D+F-l+2l')$$

(49) $$+\left[-\frac{15}{32}\gamma e^2 m-\frac{1555}{256}\gamma e^2 m^2\right]\sin(2D+F-2l)$$

(50) $$-\frac{35}{32}\gamma e^2 e' m\sin(2D+F-2l-l')$$

(51) $$+\frac{15}{32}\gamma e^2 e' m\sin(2D+F-2l+l')$$

(52) $$+\frac{15}{8}\gamma e^3 m\sin(2D+F-3l)$$

(53) $$-\frac{11}{16}\gamma^3 m^2\sin(2D+3F)$$

(54) $$-\frac{15}{8}\gamma^3 e m\sin(2D+3F-l)$$

(55) $$+\left[\left(\frac{3}{4}\gamma+\frac{9}{8}\gamma^3+\frac{27}{16}\gamma e^2-\frac{15}{8}\gamma e'^2\right)m\right.$$
$$+\left(\frac{25}{16}\gamma-\frac{175}{32}\gamma^3+\frac{423}{64}\gamma e^2-\frac{199}{16}\gamma e'^2\right)m^2$$
$$\left.+\frac{2957}{768}\gamma m^3+\frac{84703}{9216}\gamma m^4\right]\sin(2D-F)$$

(56) $$+\left[\left(\frac{7}{4}\gamma e'+\frac{21}{8}\gamma^3 e'+\frac{63}{16}\gamma e^2 e'-\frac{123}{32}\gamma e'^3\right)m+\frac{255}{32}\gamma e' m^2+\frac{3509}{128}\gamma e' m^3\right]$$
$$\times\sin(2D-F-l')$$

(57) $$+\left[\frac{51}{16}\gamma e'^2 m+\frac{2729}{128}\gamma e'^2 m^2\right]\sin(2D-F-2l')$$

(58) $$+\left[-\left(\frac{3}{4}\gamma e'+\frac{9}{8}\gamma^3 e'+\frac{27}{16}\gamma e^2 e'-\frac{3}{32}\gamma e'^3\right)m-\frac{115}{32}\gamma e' m^2-\frac{2083}{384}\gamma e' m^3\right]$$
$$\times\sin(2D-F+l')$$

(59) $$+\left[-\frac{9}{16}\gamma e'^2 m-\frac{57}{128}\gamma e'^2 m^2\right]\sin(2D-F+2l')$$

(60) $$-\frac{1}{32}\gamma e'^3 m\sin(2D-F+3l')$$

(61) $$+\left[\left(\frac{3}{4}\gamma e-3\gamma^3 e+\frac{123}{32}\gamma e^3-\frac{15}{8}\gamma e e'^2\right)m+\frac{23}{16}\gamma e m^2+\frac{2077}{768}\gamma e m^3\right]$$
$$\times\sin(2D-F+l)$$

(62) $$+\left[\frac{7}{4}\gamma e e' m+\frac{19}{2}\gamma e e' m^2\right]\sin(2D-F+l-l')$$

(63) $$+\frac{51}{16}\gamma e e'^2 m\sin(2D-F+l-2l')$$

(64) $$+\left[-\frac{3}{4}\gamma e e' m-\frac{11}{2}\gamma e e' m^2\right]\sin(2D-F+l+l')$$

(65) $$-\frac{9}{16}\gamma e e'^2 m\sin(2D-F+l+2l')$$

(66) $$+\left[\frac{27}{32}\gamma e^2 m+\frac{303}{128}\gamma e^2 m^2\right]\sin(2D-F+2l)$$

(67) $$+\frac{63}{32}\gamma e^2 e' m\sin(2D-F+2l-l')$$

(68) $$-\frac{27}{32}\gamma e^2 e' m\sin(2D-F+2l+l')$$

(69) $$+\gamma e^3 m\sin(2D-F+3l)$$

(70) $$+\left[\left(3\gamma e-\frac{27}{8}\gamma^3 e-\frac{3}{2}\gamma e^3-\frac{15}{2}\gamma e e'^2\right)m+\frac{105}{8}\gamma e m^2+\frac{3681}{64}\gamma e m^3\right]$$
$$\times\sin(2D-F-l)$$

(71) $$+\left[7\gamma e e' m+\frac{171}{4}\gamma e e' m^2\right]\sin(2D-F-l-l')$$

(72) $$+\frac{51}{4}\gamma e e'^2 m\sin(2D-F-l-2l')$$

(73) $$+\left[-3\gamma e e' m-\frac{3}{2}\gamma e e' m^2\right]\sin(2D-F-l+l')$$

(74) $$-\frac{9}{4}\gamma e e'^2 m\sin(2D-F-l+2l')$$

(75) $$+\left[\frac{147}{32}\gamma e^2 m+\frac{3257}{128}\gamma e^2 m^2\right]\sin(2D-F-2l)$$

(76) $$+\frac{343}{32}\gamma e^2 e' m\sin(2D-F-2l-l')$$

(77) $$-\frac{147}{32}\gamma e^2 e' m\sin(2D-F-2l+l')$$

(78) $$+\frac{67}{8}\gamma e^3 m\sin(2D-F-3l)$$

(79) $$+\left[\frac{15}{8}\gamma^3 m-\frac{91}{32}\gamma^3 m^2\right]\sin(2D-3F)$$

(80) $$+\frac{35}{8}\gamma^3 e' m\sin(2D-3F-l')$$

(81) $-\frac{15}{8}\gamma^3 e' m \sin(2D - 3F + l')$

(82) $-\frac{33}{8}\gamma^3 e m \sin(2D - 3F + l)$

(83) $+\frac{21}{4}\gamma^3 e m \sin(2D - 3F - l)$

(84) $+\frac{161}{128}\gamma m^4 \sin(4D + F)$

(85) $+\frac{105}{16}\gamma e m^3 \sin(4D + F - l)$

(86) $+\frac{2025}{256}\gamma e^2 m^2 \sin(4D + F - 2l)$

(87) $+\left[\left(-\frac{9}{64}\gamma^3 + \frac{405}{128}\gamma e^2\right)m^2 + \frac{33}{64}\gamma m^3 + \frac{621}{256}\gamma m^4\right]\sin(4D - F)$

(88) $+\frac{385}{128}\gamma e' m^3 \sin(4D - F - l')$

(89) $-\frac{99}{128}\gamma e' m^3 \sin(4D - F + l')$

(90) $+\frac{21}{16}\gamma e m^3 \sin(4D - F + l)$

(91) $+\left[\frac{45}{32}\gamma e m^2 + \frac{267}{32}\gamma e m^3\right]\sin(4D - F - l)$

(92) $+\frac{105}{16}\gamma e e' m^2 \sin(4D - F - l - l')$

(93) $-\frac{45}{16}\gamma e e' m^2 \sin(4D - F - l + l')$

(94) $+\frac{585}{256}\gamma e^2 m^2 \sin(4D - F - 2l)$

(95) $+\frac{45}{64}\gamma^3 m^2 \sin(4D - 3F)$

(96) $+\left[-\frac{15}{8}\gamma m - \frac{83}{8}\gamma m^2\right]\frac{a}{a'}\sin(D + F)$

(97) $+\frac{15}{8}\gamma e' m \frac{a}{a'}\sin(D + F - l')$

(98) $+\left[\frac{5}{2}\gamma e' - \frac{45}{4}\gamma e' m\right]\frac{a}{a'}\sin(D + F + l')$

(99) $-\frac{135}{32}\gamma e m \frac{a}{a'}\sin(D + F + l)$

(100) $+\frac{45}{8}\gamma e e' \frac{a}{a'}\sin(D + F + l + l')$

(101) $+\frac{45}{32}\gamma e m \frac{a}{a'}\sin(D + F - l)$

(102) $-\frac{5}{8}\gamma e e' \frac{a}{a'}\sin(D + F - l + l')$

(103) $+\left[-\frac{15}{8}\gamma m - \frac{411}{64}\gamma m^2\right]\frac{a}{a'}\sin(D - F)$

(104) $$+\frac{15}{16}\gamma e'm\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}-l')$$

(105) $$+\left[\frac{5}{2}\gamma e'-\frac{45}{4}\gamma e'm\right]\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}+l')$$

(106) $$-\frac{195}{32}\gamma em\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}+l)$$

(107) $$+\frac{55}{24}\gamma ee'\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}+l+l')$$

(108) $$-\frac{45}{32}\gamma em\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}-l)$$

(109) $$+\frac{25}{8}\gamma ee'\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}-l+l')$$

(110) $$+\frac{15}{32}\gamma m^2\frac{a}{a'}\sin(3\mathrm{D}+\mathrm{F})$$

(111) $$-\frac{95}{64}\gamma m^2\frac{a}{a'}\sin(3\mathrm{D}-\mathrm{F})$$

(112) $$+\frac{15}{16}\gamma e'm\frac{a}{a'}\sin(3\mathrm{D}-\mathrm{F}+l')$$

(113) $$-\frac{25}{16}\gamma em\frac{a}{a'}\sin(3\mathrm{D}-\mathrm{F}-l).$$

(1) $$\frac{1}{r}=\frac{1}{a}\Big\{1+\frac{1}{6}m^2$$

(2) $$+e\cos l$$

(3) $$-\frac{5}{2}\gamma^2 e\cos(2\mathrm{F}-l)\Big\}.$$

The new terms, which arise from the substitutions, are given in the following expressions. In the manner of DELAUNAY, the terms, arising from each operation in each term of the foregoing expressions for the three co-ordinates of the moon, are written separately. The indications beneath the lines denote the source of the terms, the first number being that of the operation, the second that of the term in the preceding expressions. The arrangement of the terms is the same as that of R given in Chapter I.

$$\mathrm{V}= \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

(1) $$+\frac{\beta_1}{a^3}\Big\{\left[-\frac{21}{4}e'm-\frac{21}{4}e'm+\frac{21}{4}e'm+\frac{21}{4}e'm-6e'm\right]\sin l'$$

[1........4] [1........6] [2........3] [3... ....3] [102..1]

(2) $$+\left[\frac{7}{2}e+\frac{5}{2}e-3e+\frac{225}{32}em-3e-\frac{225}{32}em\right]\sin l$$

[1..0] [1..8] [4..3] [15.....40] [102..............3]

(3) $$+\left[\frac{1}{2}e^2+\frac{39}{12}e^2+3e^2-\frac{53}{12}e^2+\frac{5}{3}\frac{\gamma^2e^2}{m^2}-\frac{15}{4}e^2\right]\sin 2l$$

[1..3] [1....11] [4...0] [5....3] [9.....13] [102...8]

(4) $$+\left[-2\gamma^2-3\gamma^2+4\gamma^2-\frac{14}{3}\gamma^2+7\gamma^2+\frac{25}{3}\frac{\gamma^2e^2}{m^2}-\gamma^2\right]\sin 2\mathrm{F}$$

[1.....18] [1..23] [6....0] [7......3] [8....3] [9........8] [102..13]

(5) $$+\left[\frac{20}{3}\frac{\gamma^2 e}{m^2} - \frac{105}{2}\frac{\gamma^2 e}{m}\right]\sin(2F - l)$$
[9.....................3]

(6) $$-\frac{55}{3}\frac{\gamma^2 e^2}{m^2}\sin(2F - 2l)$$
[9......0]

(7) $$+\left[\frac{17}{8}m^2 + \frac{15}{4}m + \frac{263}{16}m^2 - \frac{3}{2}m^2 - \frac{49}{24}m^2 - \frac{15}{4}m - \frac{339}{16}m^2 + \frac{75}{16}e^2\right.$$
[1...36] [1..................40] [10....0] [11....3] [12...................3] [15....8]

$$\left. + \frac{3}{4}\gamma^2 + \frac{11}{2}m^2\right]\sin 2D$$
[19...13] [102...31]

(8) $$+\left[\frac{35}{4}e'm - \frac{35}{4}e'm\right]\sin(2D - l')$$
[1....41] [13... ..3]

(9) $$+\left[-\frac{15}{4}e'm + \frac{15}{4}e'm\right]\sin(2D + l')$$
[1........43] [14.......3]

(10) $$+\left[\frac{75}{16}em + \frac{45}{8}em - \frac{45}{8}em - \frac{75}{16}em\right]\sin(2D + l)$$
[1...31] [4......40] [10......3] [12.....8]

(11) $$+\left[\frac{75}{16}em + \frac{45}{8}em - \frac{45}{8}em + \frac{15}{16}em + \frac{15}{4}e + \frac{19}{2}em + \frac{15}{8}em\right]\sin(2D - l)$$
[1....31] [1.....45] [10......3] [12.....0] [15................3] [102....40]

(12) $$+\frac{35}{6}ee'\sin(2D - l - l')$$
[16.....3]

(13) $$-\frac{15}{2}ee'\sin(2D - l + l')$$
[17.. ..3]

(14) $$+\frac{15}{4}\frac{ee'^2}{m}\sin(2D - l + 2l')$$
[18.....3]

(15) $$-\frac{15}{4}e^2\sin(2D - 2l)$$
[15....0]

(16) $$+3\gamma^2\sin(2D - 2F)$$
[19...0]

(17) $$+\frac{25}{2}\frac{\gamma^2 e}{m}\sin(2D - 2F + l)$$
[9.....40]

(18) $$-\frac{15}{8}\frac{a}{a'}\sin D$$
[23....3]

(19) $$-\frac{1}{m^2}\frac{a}{a'}\left[\frac{10}{3}e' - \frac{185}{4}e'm\right]\sin(D + l')$$
[24.............................3]

(20) $$-\frac{25}{6}\frac{ee'}{m^2}\frac{a}{a'}\sin(\mathrm{D}+l+l')$$
[24........8]

(21) $$-\frac{25}{4}\frac{e'}{m}\frac{a}{a'}\sin(\mathrm{D}-l')$$
[24.......40]

(22) $$+\frac{25}{2}\frac{ee'}{m^2}\frac{a}{a'}\sin(\mathrm{D}-l+l')\Big\}$$
[24........0]

(23) $$+\frac{\beta_2}{a^2}\Big\{\Big[\frac{7}{4}\gamma-\frac{7}{3}\gamma+\gamma+\Big(\frac{25}{18}-\frac{475}{36}m\Big)\frac{\gamma e^2}{m^2}+\Big(\frac{2}{3}\gamma-\frac{37}{3}\gamma^3+\frac{3}{2}\gamma e^2-\gamma e'^2\Big)\frac{1}{m^2}$$
[25..0] [28..3] [30..3] [31...... ....8] [32..

$$+\Big(\frac{1}{4}\gamma-\frac{33}{8}\gamma^3-\frac{77}{4}\gamma e^2-\frac{1}{9}\gamma e'^2\Big)\frac{1}{m}+\frac{11}{36}\gamma+\frac{17675}{2304}\gamma m+\frac{8}{9}\frac{\gamma}{m^2}\frac{f}{n}$$
....13]

$$+\frac{9}{256}\gamma m\Big]\sin(\zeta+\mathrm{F})$$
[52 .....31]

(24) $$+\Big[\Big(\frac{1}{2}-\frac{19}{8}m\Big)\frac{\gamma e'}{m}+\frac{9}{16}\gamma e'\Big]\sin(\zeta+\mathrm{F}-l')$$
[32.....................14] [33......13]

(25) $$+\frac{3}{8}\frac{\gamma e'^2}{m}\sin(\zeta+\mathrm{F}-2l')$$
[32.....15]

(26) $$+\Big[-\Big(\frac{1}{2}+\frac{35}{8}m\Big)\frac{\gamma e'}{m}+\frac{9}{16}\gamma e'\Big]\sin(\zeta+\mathrm{F}+l')$$
[32.........................16] [35......13]

(27) $$-\frac{3}{8}\frac{\gamma e'^2}{m}\sin(\zeta+\mathrm{F}+2l')$$
[32....17]

(28) $$+\Big[\frac{7}{2}\gamma e+\frac{14}{3}\gamma e-\frac{17}{2}\gamma e+\frac{5}{4}\gamma e+\frac{65}{36}\frac{\gamma e^3}{m^2}+\Big(\frac{4}{3}\gamma e-\frac{114}{3}\gamma^3 e+\frac{11}{12}\gamma e^3-2\gamma ee'^2$$
[25......3] [28.....0] [29.... 3] [30... 8] [31....11] [32.. ....

$$+\frac{1}{2}\gamma em+\frac{10}{9}\gamma em^2\Big)\frac{1}{m^2}-\frac{3}{4}\gamma e\Big]\sin(\zeta+\mathrm{F}+l)$$
...............................18] [37...13]

(29) $$+\frac{9}{2}\frac{\gamma ee'}{m}\sin(\zeta+\mathrm{F}+l-l')$$
[32.....19]

(30) $$-\frac{9}{2}\frac{\gamma ee'}{m}\sin(\zeta+\mathrm{F}+l+l')$$
[32.....20]

(31) $$+\Big[\Big(\frac{13}{6}\gamma e^2+\frac{13}{16}\gamma e^2 m\Big)\frac{1}{m^2}\Big]\sin(\zeta+\mathrm{F}+2l)$$
[32................ ....21]

(32) $$+\frac{59}{18}\frac{\gamma e^3}{m^2}\sin(\zeta+\mathrm{F}+3l)$$
[32.....22]

(33) $$+\Big[\frac{9}{2}\gamma e - \frac{35}{12}\gamma e - 2\gamma e + \Big(\frac{10}{9} - \frac{77}{36}e^2 - \frac{5}{3}e'^2 - \frac{95}{9}m + \frac{3989}{108}m^2\Big)\frac{\gamma e}{m^2}$$

[25.....3] [28.....8] [30...0] [31..3]

$$+\Big(2 - 2\gamma^2 - \frac{61}{12}e^2 - 3e'^2 - \frac{21}{2}m - \frac{1}{48}m^2\Big)\frac{\gamma e}{m^2} + \frac{3}{4}\gamma e\Big]\sin(\zeta + F - l)$$

[32.. .... .23] [39...13]

(34) $$+\Big[\frac{35}{12}\frac{\gamma e e'}{m} - \frac{15}{4}\frac{\gamma e e'}{m}\Big]\sin(\zeta + F - l - l')$$

[31.........4] [32.......24]

(35) $$+\Big[-\frac{35}{12}\frac{\gamma e e'}{m} + \frac{15}{4}\frac{\gamma e e'}{m}\Big]\sin(\zeta + F - l + l')$$

[31............6] [32......25]

(36) $$+\Big[-\Big(\frac{35}{12} - \frac{1085}{48}m\Big)\frac{\gamma e^2}{m^2} - \Big(\frac{1}{3} + \frac{23}{4}m\Big)\frac{\gamma e^2}{m^2}\Big]\sin(\zeta + F - 2l)$$

[31....... ....0] [32..... .. ..26]

(37) $$+\Big[-\frac{58}{27}\frac{\gamma e^3}{m^2} - \frac{7}{9}\frac{\gamma e^3}{m^2}\Big]\sin(\zeta + F - 3l)$$

[31............3] [32...27]

(38) $$-\Big(\frac{2}{3} + \frac{1}{4}m\Big)\frac{\gamma^3}{m^2}\sin(\zeta + 3F)$$

[32.................28]

(39) $$-\frac{8}{3}\frac{\gamma^3 e}{m^2}\sin(\zeta + 3F + l)$$

[32...29]

(40) $$+\Big[-\frac{10}{9}\frac{\gamma^3 e}{m^2} - 4\frac{\gamma^3 e}{m^2}\Big]\sin(\zeta + 3F - l)$$

[31..........18] [32....30]

(41) $$+\Big[\frac{1}{4}\gamma + \Big(\frac{38}{3} - 7\gamma^2 - \frac{20}{3}e^2 - 19e'^2 + \Big(\frac{13}{4} - \frac{135}{8}\gamma^2 - 88e^2 - \frac{13}{9}e'^2\Big)m$$

[25...13] [32... ....

$$+\frac{135513}{288}m^2 + \frac{5825}{576}m^3 + \frac{152}{9}\frac{1}{m^2}\frac{f}{n}\Big)\frac{\gamma}{m^2} + 3\gamma - 3\gamma\Big]\sin(\zeta - F)$$

....0] [37..3] [39..3]

(42) $$+\Big[\Big(\frac{9}{2} - \frac{51}{16}m\Big)\frac{\gamma e'}{m} + \frac{63}{16}\gamma e'\Big]\sin(\zeta - F - l')$$

[32....... ....1] [33......0]

(43) $$+\frac{27}{8}\frac{\gamma e'^2}{m}\sin(\zeta - F - 2l')$$

[32.......2]

(44) $$+\Big[-\Big(\frac{9}{2} - \frac{51}{16}m\Big)\frac{\gamma e'}{m} + \frac{63}{16}\gamma e'\Big]\sin(\zeta - F + l')$$

[32........... ....1] [35......0]

(45) $$-\frac{27}{8}\frac{\gamma e'^2}{m}\sin(\zeta - F + 2l')$$

[32.......2]

(46) $$+\left[\frac{3}{4}\gamma e+\frac{7}{12}\gamma e+\left(\frac{40}{3}+\frac{135}{6}\gamma^2-\frac{80}{3}e^2-20e'^2+\frac{53}{2}m+\frac{6115}{36}m^2\right)\frac{\gamma e}{m^2}-6\gamma e\right.$$

[25....23] [28....13] [32.............................3] [37...0

$$\left.+\frac{13}{4}\gamma e-\frac{15}{4}\gamma e\right]\sin(\zeta-\mathrm{F}+l)$$

[38......3] [39....8]

(47) $$+\frac{91}{2}\frac{\gamma ee'}{m}\sin(\zeta-\mathrm{F}+l-l')$$

[32.......4]

(48) $$-\frac{91}{2}\frac{\gamma ee'}{m}\sin(\zeta-\mathrm{F}+l+l')$$

[32 ......6]

(49) $$+\left(\frac{205}{12}+\frac{255}{8}m\right)\frac{\gamma e^2}{m^2}\sin(\zeta-\mathrm{F}+2l)$$

[32... ....8]

(50) $$+\frac{45}{2}\frac{\gamma e^3}{m^2}\sin(\zeta-\mathrm{F}+3l)$$

[32......11]

(51) $$+\left[\frac{1}{2}\gamma e-\frac{1}{4}\gamma e+\frac{5}{3}\frac{\gamma^3 e}{m^2}-\frac{5}{12}\frac{\gamma e^3}{m^2}+\left(\frac{40}{3}+\frac{235}{6}\gamma^2-\frac{115}{4}e^2-20e'^2+\frac{53}{2}m\right.\right.$$

[25.. 18] [30...13] [31.................23] [32..............................

$$\left.\left.+\frac{1360}{9}m^2\right)\frac{\gamma e}{m^2}+\frac{15}{4}\gamma e+6\gamma e-\frac{9}{2}\gamma e\right]\sin(\zeta-\mathrm{F}-l)$$

.....................3] [37... .8] [39...0] [40....3]

(52) $$-\frac{49}{2}\frac{\gamma ee'}{m}\sin(\zeta-\mathrm{F}-l-l')$$

[32 ......6]

(53) $$+\frac{49}{2}\frac{\gamma ee'}{m}\sin(\zeta-\mathrm{F}-l+l')$$

[32.......4]

(54) $$+\left[\left(-\frac{5}{36}+\frac{95}{72}m\right)\frac{\gamma e^2}{m^2}+\left(\frac{65}{4}+\frac{275}{8}m\right)\frac{\gamma e^2}{m^2}\right]\sin(\zeta-\mathrm{F}-2l)$$

[31...........................13] [32.......................8]

(55) $$+\left[-\frac{5}{18}\frac{\gamma e^3}{m^2}+\frac{125}{6}\frac{\gamma e^3}{m^2}\right]\sin(\zeta-\mathrm{F}-3l)$$

[31...........18] [32......11]

(56) $$+\left(-\frac{40}{3}-\frac{7}{2}m\right)\frac{\gamma^3}{m^2}\sin(\zeta-3\mathrm{F})$$

[32.......................13]

(57) $$+\left[-25\frac{\gamma^3 e}{m^2}+\frac{23}{3}\frac{\gamma^3 e}{m^2}\right]\sin(\zeta-3\mathrm{F}+l)$$

[32...........23] [42......3]

(58) $$-40\frac{\gamma^3 e}{m^2}\sin(\zeta-3\mathrm{F}-l)$$

[32.... 18]

(59) $$+\left[-\frac{3}{32}\gamma m+\frac{35}{8}\gamma m-\left(\frac{3}{4}\gamma^2-\frac{65}{8}e^2-\frac{11}{12}m-\frac{1043}{288}m^2\right)\frac{\gamma}{m}\right.$$

[25...........31] [28......40] [32...................................49

$$\left.-\frac{35}{8}\gamma m+\frac{3}{32}\gamma m\right]\sin(\zeta+2\mathrm{D}+\mathrm{F})$$

[44......3] [45 ......12]

(60) $$+\frac{77}{24}\gamma e' \sin(\zeta + 2D + F - l')$$
[32.....50]

(61) $$-\frac{11}{24}\gamma e' \sin(\zeta + 2D + F + l')$$
[32.....51]

(62) $$+\frac{13}{4}\gamma e \sin(\zeta + 2D + F + l)$$
[32.....52]

(63) $$+\left[\frac{85}{72}\gamma e + \left(\frac{5}{2} + \frac{653}{48}m\right)\frac{\gamma e}{m}\right] \sin(\zeta + 2D + F - l)$$
[31.....36] [32..................53]

(64) $$+\frac{35}{6}\frac{\gamma e e'}{m} \sin(\zeta + 2D + F - l - l')$$
[32......54]

(65) $$-\frac{5}{2}\frac{\gamma e e'}{m} \sin(\zeta + 2D + F - l + l')$$
[32.....55]

(66) $$+\left[\frac{85}{32}\frac{\gamma e^2}{m} + 5\frac{\gamma e^2}{m}\right] \sin(\zeta + 2D + F - 2l)$$
[31.......31] [32....56]

(67) $$+\left[\frac{9}{16}\gamma m + \left(\frac{1}{4} - \frac{65}{8}\gamma^2 + 62e^2 - e'^2 + \frac{1775}{96}m + \frac{161627}{2304}m^2\right)\frac{\gamma}{m} - \frac{45}{8}\gamma m\right.$$
[25.......57] [32..31] [37.....40]

$$\left. + \frac{3}{32}\gamma m + 6\gamma m - \frac{7}{8}\gamma m\right] \sin(\zeta + 2D - F)$$
[45.......0] [49....3] [48.....3

(68) $$+\left(\frac{7}{12} + \frac{6291}{96}m\right)\frac{\gamma e'}{m} \sin(\zeta + 2D - F - l')$$
[32.........................32]

(69) $$+\frac{17}{16}\frac{\gamma e'^2}{m} \sin(\zeta + 2D - F - 2l')$$
[32......33]

(70) $$-\left(\frac{1}{4} + \frac{991}{96}m\right)\frac{\gamma e'}{m} \sin(\zeta + 2D - F + l')$$
[32.......................34]

(71) $$-\frac{3}{16}\frac{\gamma e'^2}{m} \sin(\zeta + 2D - F + 2l')$$
[32......35]

(72) $$+\left(\frac{1}{2} + \frac{2063}{48}m\right)\frac{\gamma e}{m} \sin(\zeta + 2D - F + l)$$
[32.......................36]

(73) $$+\frac{7}{6}\frac{\gamma e e'}{m} \sin(\zeta + 2D - F + l - l')$$
[32.....37]

(74) $$-\frac{1}{2}\frac{\gamma e e'}{m} \sin(\zeta + 2D - F + l + l')$$
[32.....38]

(75) $$+\frac{13}{16}\frac{\gamma e^2}{m} \sin(\zeta + 2D - F + 2l)$$
[32.....39]

$$+\left[\left(\frac{49}{2}+\frac{461}{6}m\right)\frac{\gamma e}{m}-\frac{15}{2}\gamma e\right]\sin(\zeta+2D-F-l) \tag{76}$$
[32.........................40] [50.....3]

$$+\frac{343}{6}\frac{\gamma ee'}{m}\sin(\zeta+2D-F-l-l') \tag{77}$$
[32........41]

$$-\frac{49}{2}\frac{\gamma ee'}{m}\sin(\zeta+2D-F-l+l') \tag{78}$$
[32......43]

$$+\left[-\frac{11}{8}\frac{\gamma e^2}{m}-\frac{5}{16}\frac{\gamma e^2}{m}\right]\sin(\zeta+2D-F-2l) \tag{79}$$
[32...........45] [31.....57]

$$-\frac{1}{4}\frac{\gamma^3}{m}\sin(\zeta+2D-3F) \tag{80}$$
[32...57]

$$+\left[\frac{3}{32}\gamma m-\frac{15}{8}\gamma m-\frac{25}{8}\frac{\gamma e^2}{m}+\left(\frac{3}{2}-3\gamma^2-\frac{25}{4}e^2-6e'^2-\frac{149}{48}m+\frac{1021}{1152}m^2\right)\frac{\gamma}{m}\right. \tag{81}$$
[25.......31] [30.....40] [31......45] [32..... ....57]

$$\left.-\left(\frac{21}{32}-\frac{11}{256}m\right)\gamma+3\gamma m-\frac{9}{8}\gamma m\right]\sin(\zeta-2D+F)$$
[52......................0] [57.....3] [58.... 3]

$$-\left(\frac{3}{2}+\frac{263}{48}m\right)\frac{\gamma e'}{m}\sin(\zeta-2D+F-l') \tag{82}$$
[32. ....60]

$$+\left[-\frac{9}{8}\frac{\gamma e'^2}{m}-\frac{39}{16}\frac{\gamma e'^2}{m}\right]\sin(\zeta-2D+F-2l') \tag{83}$$
[32..... ....61] [54.......0]

$$+\left[\left(\frac{7}{2}-\frac{289}{48}m\right)\frac{\gamma e'}{m}-\frac{49}{48}\gamma e'\right]\sin(\zeta-2D+F+l') \tag{84}$$
[32.......................58] [55.......0]

$$+\frac{51}{8}\frac{\gamma e'^2}{m}\sin(\zeta-2D+F+2l') \tag{85}$$
[32......59]

$$+\left[\left(1-\frac{235}{24}m\right)\frac{\gamma e}{m}+\frac{3}{4}\gamma e\right]\sin(\zeta-2D+F+l) \tag{86}$$
[32.................. ...66] [52....3]

$$-\frac{\gamma ee'}{m}\sin(\zeta-2D+F+l-l') \tag{87}$$
[32..68]

$$+\frac{7}{3}\frac{\gamma ee'}{m}\sin(\zeta-2D+F+l+l') \tag{88}$$
[32....67]

$$-\frac{5}{4}\frac{\gamma e^2}{m}\sin(\zeta-2D+F+2l) \tag{89}$$
[32....69]

$$+\left[-\left(\frac{25}{12}-\frac{1535}{144}m\right)\frac{\gamma e}{m}-\left(\frac{11}{4}-\frac{11}{8}m\right)\frac{\gamma e}{m}+\frac{3}{4}\gamma e\right]\sin(\zeta-2D+F-l) \tag{90}$$
[31.............. ....40] [32...................62] [52.....3]

(91) $$+\left[\frac{25}{12}\frac{\gamma ee'}{m}+\frac{11}{4}\frac{\gamma ee'}{m}\right]\sin(\zeta-2D+F-l-l')$$
[31 … 43] [32 … 64]

(92) $$-\left[\frac{175}{36}\frac{\gamma ee'}{m}+\frac{77}{12}\frac{\gamma ee'}{m}\right]\sin(\zeta-2D+F-l+l')$$
[31 … 41] [32 … 63]

(93) $$-\left[\frac{85}{32}\frac{\gamma e^2}{m}+\frac{15}{4}\frac{\gamma e^2}{m}\right]\sin(\zeta-2D+F-2l)$$
[31 … 31] [32 … 65]

(94) $$-2\frac{\gamma^3}{m}\sin(\zeta-2D+3F)$$
[32 … 70]

(95) $$+\left[\left(-\frac{1}{4}-\frac{87}{8}\gamma^2+\frac{227}{4}e^2+e'^2+\frac{523}{32}m+\frac{145941}{2304}m^2\right)\frac{\gamma}{m}+\frac{45}{8}\gamma m\right.$$
[32 … 31] [39 … 40]

$$\left.-\left(\frac{3}{32}-\frac{11}{256}m\right)\gamma-\frac{45}{8}\gamma m\right]\sin(\zeta-2D-F)$$
[52 … 13] [60 … 3]

(96) $$+\left(\frac{1}{4}-\frac{681}{96}m\right)\frac{\gamma e'}{m}\sin(\zeta-2D-F-l')$$
[32 … 34]

(97) $$+\left[\frac{3}{16}\frac{\gamma e'^2}{m}+\frac{3}{16}\frac{\gamma e'^2}{m}\right]\sin(\zeta-2D-F-2l')$$
[32 … 35] [54 … 13]

(98) $$+\left[-\left(\frac{7}{12}-\frac{5413}{96}m\right)\frac{\gamma e'}{m}-\frac{7}{48}\gamma e'\right]\sin(\zeta-2D-F+l')$$
[32 … 32] [55 … 13]

(99) $$-\frac{17}{16}\frac{\gamma e'^2}{m}\sin(\zeta-2D-F+2l')$$
[32 … 33]

(100) $$+\left[\left(\frac{41}{2}+\frac{545}{12}m\right)\frac{\gamma e}{m}-\frac{9}{32}\gamma e+\frac{195}{32}\gamma e\right]\sin(\zeta-2D-F+l)$$
[32 … 40] [52 … 23] [61 … 3]

(101) $$-\frac{41}{2}\frac{\gamma ee'}{m}\sin(\zeta-2D-F+l-l')$$
[32 … 43]

(102) $$+\frac{287}{6}\frac{\gamma ee'}{m}\sin(\zeta-2D-F+l+l')$$
[32 … 41]

(103) $$-\frac{19}{8}\frac{\gamma e^2}{m}\sin(\zeta-2D-F+2l)$$
[32 … 45]

(104) $$-\left[\left(\frac{1}{2}-\frac{627}{16}m\right)\frac{\gamma e}{m}+\frac{3}{16}\gamma e\right]\sin(\zeta-2D-F-l)$$
[32 … 36] [52 … 18]

(105) $$+\frac{1}{2}\frac{\gamma ee'}{m}\sin(\zeta-2D-F-l-l')$$
[32 … 38]

(106) $$-\frac{7}{6}\frac{\gamma e e'}{m}\sin(\zeta - 2D - F - l + l')$$
[32.......37]

(107) $$-\frac{13}{16}\frac{\gamma e^2}{m}\sin(\zeta - 2D - F - 2l)$$
[32. ....39]

(108) $$+\frac{1}{4}\frac{\gamma^3}{m}\sin(\zeta - 2D - 3F)$$
[32...49]

(109) $$+\frac{11}{32}\gamma m\sin(\zeta + 4D - F)$$
[32.....71]

(110) $$+\frac{15}{16}\gamma e\sin(\zeta + 4D - F - l)$$
[32 ...72]

(111) $$-\left[\left(\frac{3}{32} - \frac{331}{256}m\right)\gamma + \frac{9}{256}\gamma m\right]\sin(\zeta - 4D + F)$$
[32......................74] [52.......31]

(112) $$+\frac{3}{16}\gamma e'\sin(\zeta - 4D + F - l')$$
[32.....76]

(113) $$-\frac{7}{16}\gamma e'\sin(\zeta - 4D + F + l')$$
[32.....75]

(114) $$+\frac{33}{16}\gamma e\sin(\zeta - 4D + F + l)$$
[32....78]

(115) $$-\frac{3}{16}\gamma e\sin(\zeta - 4D + F - l)$$
[32.....77]

(116) $$-\frac{11}{32}\gamma m\sin(\zeta - 4D - F)$$
[32......71]

(117) $$-\frac{15}{16}\gamma e\sin(\zeta - 4D - F + l)$$
[32.....72]

(118) $$-\frac{5}{4}\frac{\gamma}{m}\frac{a}{a'}\sin(\zeta + D + F)$$
[32.....85]

(119) $$+\frac{5}{3}\frac{\gamma e'}{m^3}\frac{a}{a'}\sin(\zeta + D + F + l')$$
[32......86]

(120) $$-\frac{25}{117}\frac{\gamma e e'}{m^3}\frac{a}{a'}\sin(\zeta + D + F - l + l')$$
[62.... .... .13]

(121) $$-\frac{75}{4}\frac{\gamma}{m}\frac{a}{a'}\sin(\zeta + D - F)$$
[32......79]

(122) $$+\left[\frac{55}{3}\frac{\gamma e'}{m^2}\frac{a}{a'}+\left(\frac{100}{117}+\frac{800}{1521}\frac{\gamma^2}{m}+\frac{2000}{4563}\frac{e^2}{m}-\frac{17710}{4563}m\right)\frac{\gamma e'}{m^3}\frac{a}{a'}\right]\sin(\zeta+D-F+l')$$
[32.........80] [62..........3]

(123) $$+\frac{125}{117}\frac{\gamma ee'}{m^3}\frac{a}{a'}\sin(\zeta+D-F+l+l')$$
[62...........8]

(124) $$-\frac{550}{117}\frac{\gamma ee'}{m^3}\frac{a}{a'}\sin(\zeta+D-F-l+l')$$
[62...........0]

(125) $$+\frac{1000}{4563}\frac{\gamma e^2 e'}{m^4}\frac{a}{a'}\sin(\zeta+D-F-2l+l')$$
[62.... ....3]

(126) $$-\frac{25}{4}\frac{\gamma}{m}\frac{a}{a'}\sin(\zeta-D+F)$$
[32......87]

(127) $$+\left[\frac{5}{9}\frac{\gamma e'}{m^2}\frac{a}{a'}+\frac{10}{9}\frac{\gamma e'}{m^2}\frac{a}{a'}\right]\sin(\zeta-D+F-l')$$
[32.........88] [63.........3]

(128) $$-5\frac{\gamma}{m}\frac{a}{a'}\sin(\zeta-D-F)$$
[32....79]

(129) $$+\left[\frac{40}{3}\frac{\gamma e'}{m^2}\frac{a}{a'}-\frac{5}{3}\frac{\gamma e'}{m^2}\frac{a}{a'}\right]\sin(\zeta-D-F-l')$$
[32..........80] [64........3]

(130) $$-\frac{125}{78}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(\zeta-D-F+l')$$
[62..........40]

(131) $$-\frac{25}{12}\frac{\gamma}{m}\frac{a}{a'}\sin(\zeta-3D+F)\Big\}$$
[32......89]

(132) $$+\frac{\beta_3}{a^3}\Big\{\Big[\frac{3}{4}-2\gamma^2-\frac{5}{2}e^2+\frac{17}{8}m^2-\frac{7}{6}+\frac{7}{3}\gamma^2+\frac{107}{12}e^2-\frac{8773}{1152}m^2-\frac{85}{16}e^2$$
[65..........0] [68.......... ....3] [71....8]

$$+\frac{1}{2}-\gamma^2-\frac{5}{4}e^2+\frac{1555}{384}m^2-\frac{25}{6}\frac{\gamma^2e^2}{m^2}-\frac{5}{48}e^2-\Big(\frac{2}{3}-\frac{14}{3}\gamma^2+\frac{3}{2}e^2-e'^2+\frac{1}{4}m$$
[73..........3] [76..........8] [79.......... ....

$$+\frac{7}{18}m^2\Big)\frac{\gamma^2}{m^2}+\frac{525}{128}m^2-\frac{225}{128}m^2\Big]\sin 2\zeta$$
..........13] [86 .....40] [93......40]

(133) $$+\left[\frac{9}{4}e'm+\frac{49}{16}e'm-\frac{105}{16}e'm+\frac{21}{16}e'm+\frac{3}{16}e'm-\frac{1}{2}\frac{\gamma^2e'}{m}\right]\sin(2\zeta-l')$$
[65. .....1] [68.......6] [69.......3] [73.......4] [74.......3] [79......14]

(134) $$+\left[-\frac{9}{4}e'm-\frac{49}{16}e'm+\frac{105}{16}e'm-\frac{21}{16}e'm-\frac{3}{16}e'm+\frac{1}{2}\frac{\gamma^2e'}{m}\right]\sin(2\zeta+l')$$
[65...........1] [68.......4] [70........3] [73... ...6] [75.......3] [79 ....16]

(135) $$+\left[\frac{7}{4}e+\frac{49}{24}e-\frac{17}{4}e+\frac{5}{8}e-\left(\frac{4}{3}+\frac{1}{2}m\right)\frac{\gamma^2e}{m^2}\right]\sin(2\zeta+l)$$
[65...3] [68...0] [71...3] [73...8] [79..........18]

$$+\left[\frac{35}{16}e^2+\frac{43}{12}e^2+\frac{17}{4}e^2-\frac{169}{16}e^2+\frac{13}{16}e^2-\frac{13}{6}\frac{\gamma^2e^2}{m^2}\right]\sin(2\zeta+2l) \tag{136}$$

[65......8] [68....3] [71....0] [72......3] [73....11] [79.....21]

$$+\left[\frac{9}{4}e-\frac{35}{24}e-\frac{7}{8}e-\left(\frac{10}{3}\gamma^2-\frac{85}{2}\gamma^2m+\frac{1}{12}m^2-\frac{625}{32}m^3\right)\frac{e}{m^2}\right. \tag{137}$$

[65...3] [68. .8] [73...0] [76.......................................3]

$$\left.-\left(2-\frac{21}{2}m\right)\frac{\gamma^2e}{m^2}\right]\sin(2\zeta-l)$$

[79...................23]

$$+\left[\frac{45}{16}e^2-\frac{91}{48}e^2-e^2+\frac{55}{6}\frac{\gamma^2e^2}{m^2}+\frac{1}{12}e^2+\frac{1}{16}e^2+\frac{1}{3}\frac{\gamma^2e^2}{m^2}\right]\sin(2\zeta-2l) \tag{138}$$

[65......8] [68..11] [73..3] [76.................0] [77....3] [79......26]

$$+\left[-\frac{3}{4}\gamma^2-\frac{7}{4}\gamma^2-\frac{1}{2}\gamma^2+\frac{35}{12}\gamma^2+\frac{2}{3}\frac{\gamma^4}{m^2}\right]\sin(2\zeta+2F) \tag{139}$$

[65........13] [68..23] [73...18] [78....3] [79...28]

$$+\frac{5}{12}\frac{\gamma^2e^2}{m^2}\sin(2\zeta+2F-2l) \tag{140}$$

[76......13]

$$+\left[-\frac{5}{4}\gamma^2+\frac{7}{6}\gamma^2+\frac{3}{4}\gamma^2-\left(\frac{20}{3}+\frac{22}{3}\gamma^2-\frac{10}{3}e^2-10e'^2+\frac{7}{4}m+\frac{3785}{144}m^2\right)\frac{\gamma^2}{m^2}\right. \tag{141}$$

[65.......13] [68..18] [73..23] [79.......................................0]

$$\left.-\frac{17}{4}\gamma^2+3\gamma^2\right]\sin(2\zeta-2F)$$

[82.....3] [83...3]

$$-\frac{9}{2}\frac{\gamma^2e'}{m}\sin(2\zeta-2F-l') \tag{142}$$

[79.....1]

$$+\frac{9}{2}\frac{\gamma^2e'}{m}\sin(2\zeta-2F+l') \tag{143}$$

[79.....1]

$$-\left(\frac{20}{3}+\frac{53}{4}m\right)\frac{\gamma^2e}{m^2}\sin(2\zeta-2F+l) \tag{144}$$

[79.....................3]

$$-\frac{35}{4}\frac{\gamma^2e^2}{m^2}\sin(2\zeta-2F+2l) \tag{145}$$

[79.......8]

$$-\left(\frac{20}{3}+\frac{53}{4}m\right)\frac{\gamma^2e}{m^2}\sin(2\zeta-2F-l) \tag{146}$$

[79.....................3]

$$+\left[\frac{5}{12}\frac{\gamma^2e^2}{m^2}-\frac{95}{12}\frac{\gamma^2e^2}{m^2}\right]\sin(2\zeta-2F-2l) \tag{147}$$

[76........13] [79.......8]

$$+\frac{20}{3}\frac{\gamma^4}{m^2}\sin(2\zeta-4F) \tag{148}$$

[79....13]

(149) $$+\left[\frac{99}{32}m^2+\frac{35}{16}m+\frac{1841}{192}m^2+\frac{17}{32}m^2-\frac{11}{2}\gamma^2-\frac{69}{64}m^2-\frac{93}{32}m^2\right.$$
[65......31] [68..................40] [73......36] [79....49] [84....0] [85......3]

$$\left.-\frac{35}{16}m-\frac{1753}{192}m^2\right]\sin(2\zeta+2D)$$
[86........ ....3]

(150) $$+\left[\frac{245}{48}e'm-\frac{245}{48}e'm\right]\sin(2\zeta+2D-l')$$
[68.........41] [87........3]

(151) $$+\left[-\frac{35}{16}e'm+\frac{35}{16}e'm\right]\sin(2\zeta+2D+l')$$
[68...........43] [88.......3]

(152) $$+\left[\frac{175}{64}em+\frac{255}{32}em-\frac{255}{32}em-\frac{175}{64}em\right]\sin(2\zeta+2D+l)$$
[68........31] [71......40] [84........3] [86.......8]

(153) $$+\left[\frac{45}{32}em+\frac{105}{32}em+\frac{75}{64}em-\frac{5}{2}\frac{\gamma^2 e}{m}-\frac{255}{32}em+\frac{35}{64}em+\frac{15}{8}em\right]\sin(2\zeta+2D-l)$$
[65......40] [68......45] [73......31] [79 53] [84.......3] [86......0] [89.....3]

(154) $$-\left[\frac{1}{4}+\frac{983}{96}m\right]\frac{\gamma^2}{m}\sin(2\zeta+2D-2F)$$
[79.............. ....31]

(155) $$-\frac{7}{12}\frac{\gamma^2 e'}{m}\sin(2\zeta+2D-2F-l')$$
[79......32]

(156) $$+\frac{1}{4}\frac{\gamma^2 e'}{m}\sin(2\zeta+2D-2F+l')$$
[79.....34]

(157) $$-\frac{1}{2}\frac{\gamma^2 e}{m}\sin(2\zeta+2D-2F+l)$$
[79....36]

(158) $$-\frac{29}{2}\frac{\gamma^2 e}{m}\sin(2\zeta+2D-2F-l)$$
[79.....40]

(159) $$+\left[-\frac{33}{32}m^2-\frac{119}{96}m^2-\frac{15}{16}m-\frac{263}{64}m^2-\left(\frac{3}{2}-\frac{149}{48}m\right)\frac{\gamma^2}{m}+\frac{3}{2}\gamma^2+\frac{1}{8}m^2\right.$$
[65..........31] [68.....36] [73..................40] [79.....................57] [91..............0]

$$\left.+\frac{15}{16}m+\frac{391}{64}m^2+\frac{5}{32}m^2\right]\sin(2\zeta-2D)$$
[93..................3] [97... .3]

(160) $$+\left[\frac{15}{16}e'm+\frac{3}{2}\frac{\gamma^2 e'}{m}-\frac{15}{16}e'm\right]\sin(2\zeta-2D-l')$$
[73 ......43] [79....60] [94......3]

(161) $$+\left[-\frac{35}{16}e'm-\frac{7}{2}\frac{\gamma^2 e'}{m}+\frac{35}{16}e'm\right]\sin(2\zeta-2D+l')$$
[73....... ...41] [79.....58] [95......3]

(162) $$+\left[-\frac{105}{32}em-\frac{175}{64}em-\frac{45}{32}em-\frac{\gamma^2 e}{m}-\frac{15}{64}em+\frac{15}{2}em\right]\sin(2\zeta-2D+l)$$
[65...........40] [68.......31] [73....45] [79..66] [93.....0] [96......3]

(163) $+\left[-\frac{75}{64}em+\left(\frac{25}{4}\gamma^2e+\frac{5}{32}em^2\right)\frac{1}{m}+\frac{11}{4}\frac{\gamma^2e}{m}+\frac{75}{64}em\right]\sin(2\zeta-2D-l)$

[73 .... 31] [76........................40] [79.....62] [93..... 8

(164) $+\frac{3}{16}\gamma^2\sin(2\zeta-2D+2F)$

[91....13]

(165) $+\left[\left(\frac{1}{4}-\frac{259}{32}m\right)\frac{\gamma^2}{m}+\frac{3}{16}\gamma^2\right]\sin(2\zeta-2D-2F)$

[79......................31] [91....13]

(166) $-\frac{1}{4}\frac{\gamma^2e'}{m}\sin(2\zeta-2D-2F-l')$

[79....34]

(167) $+\frac{7}{12}\frac{\gamma^2e'}{m}\sin(2\zeta-2D-2F+l')$

[79......32]

(168) $-\frac{21}{2}\frac{\gamma^2e}{m}\sin(2\zeta-2D-2F+l)$

[79.....40]

(169) $+\frac{1}{2}\frac{\gamma^2e}{m}\sin(2\zeta-2D-2F-l)$

[79...36]

(170) $+\frac{3}{32}\gamma^2\sin(2\zeta-4D)$

[79....74]

(171) $+\frac{225}{64}em\sin(2\zeta-4D+l)$

[99.......3]

(172) $+\frac{45}{2}\frac{e'}{m}\frac{a}{a'}\sin(2\zeta-D-l')\Big\}.$

[102...... ..3]

$U = \quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .$

$+\frac{\beta_1}{a^3}\Big\{\left[2\gamma-2\gamma-\gamma-\frac{9}{32}\gamma m+\gamma+\frac{9}{32}\gamma m\right]\sin F$

[1 ....8] [1..18] [6..1] [19....55] [102.. ....1]

(1) $+\left[\frac{9}{2}\gamma e+\frac{9}{2}\gamma e-3\gamma e+\gamma e-\frac{7}{3}\gamma e-2\gamma e\right]\sin(F+l)$

[1......1] [1...13] [4...18] [6...18] [7...1] [102..8]

(2) $+\left[-\frac{17}{2}\gamma e-3\gamma e+3\gamma e-\gamma e+\frac{7}{2}\gamma e-\frac{5}{3}\frac{\gamma e^3}{m^2}+\frac{20}{3}\frac{\gamma^3e}{m^2}+2\gamma e\right]\sin(F-l)$

[1............1] [1...23] [4....8] [6..8] [8....1] [9..................18] [102..18]

(3) $+\left[\frac{5}{3}\frac{\gamma e^2}{m^2}-\frac{105}{8}\frac{\gamma e^2}{m}\right]\sin(F-2l)$

[9.......... ....1]

(4) $+\frac{5}{3}\frac{\gamma e^3}{m^2}\sin(F-3l)$

[9......8]

(5) $$+\frac{20}{3}\frac{\gamma^3 e}{m^2}\sin(3F - l)$$
[9 ......8]

(6) $$+\left[\frac{15}{4}\gamma m + \frac{3}{8}\gamma m - \frac{3}{8}\gamma m - \frac{15}{4}\gamma m\right]\sin(2D + F)$$
[1........44] [6.....55] [10.....1] [12.......8]

(7) $$+\frac{15}{4}\gamma e\sin(2D + F - l)$$
[15.....8]

(8) $$-\frac{5}{8}\frac{\gamma e^2}{m}\sin(2D + F - 2l)$$
[9....55]

(9) $$+\left[\frac{3}{4}\gamma m + 3\gamma m - \frac{3}{8}\gamma m - \frac{15}{4}\gamma m - \frac{3}{4}\gamma - \gamma m + \frac{15}{8}\gamma m\right]\sin(2D - F)$$
[1......61] [1....70] [10......1] [12.....18] [19......... . 1] [102.....55]

(10) $$-\frac{7}{6}\gamma e'\sin(2D - F - l')$$
[20....1]

(11) $$+\frac{3}{2}\gamma e'\sin(2D - F + l')$$
[21....1]

(12) $$+\frac{3}{4}\frac{\gamma e'^2}{m}\sin(2D - F + 2l')$$
[22.....1]

(13) $$-\frac{3}{4}\gamma e\sin(2D - F + l)$$
[19....8]

(14) $$+\left[\frac{15}{4}\gamma e + \frac{3}{4}\gamma e\right]\sin(2D - F - l)$$
[15......18] [19..18]

(15) $$-\frac{10}{3}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(D + F + l')$$
[24.........8]

(16) $$-\frac{10}{3}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(D - F + l')\Big\}$$
[24.... ...18]

(17) $$+\frac{\beta_2}{a^2}\Big\{\Big[-\frac{1}{4} + 3\gamma^2 + \frac{3}{4}e^2 + \frac{29}{768}m^2 - \frac{7}{3}\gamma^2 - \frac{7}{12}e^2 + \gamma^2 - \frac{1}{4}e^2 + \left(\frac{5}{3}\gamma^2 - \frac{5}{48}e^2\right)\frac{e^2}{m^2}$$
[25............... ....1] [28.............8] [30. ......18] [31.......................23]

$$-\Big[\left(\frac{2}{3} - \frac{40}{3}\gamma^2 - \frac{2}{3}e^2 - e'^2 + \frac{13}{2}\gamma^4 - \frac{5}{3}\gamma^2 e^2 + \frac{257}{96}e^4 + 20\gamma^2 e'^2 + e^2 e'^2 + \frac{1}{4}e'^4\right)\frac{1}{m^2}$$
[32.. ....

$$+\left(\frac{1}{4} - \frac{9}{2}\gamma^2 - 6e^2 - \frac{1}{9}e'^2\right)\frac{1}{m} + \frac{77}{36} - \frac{12743}{288}\gamma^2 - \frac{32749}{1152}e^2 + \frac{19}{4}e'^2 + \frac{13715}{2304}m$$
....

$$+\frac{948793}{55296}m^2 - \frac{5}{4}\frac{1}{m^2}\frac{a^2}{a'^2} + \frac{8}{9}\frac{1}{m^4}\frac{f}{n} + \frac{2}{3}\frac{1}{m^3}\frac{f}{n}\Big] + 3\gamma^2 - \frac{3}{4}e^2 - 3\gamma^2 - \frac{3}{4}e^2$$
....1] [37...........18] [39............8]

$$+\frac{9}{256}m^2 - \frac{9}{256}m - \frac{117}{2048}m^2\Big]\sin\zeta$$
[45......55] [52............... ,...55]

(18)
$$+\left[\frac{3}{32}e'm-\frac{15}{32}e'm-\left(\frac{1}{4}-11\gamma^2-\frac{5}{8}e^2-\frac{3}{32}e'^2+\frac{3}{16}m-\frac{8213}{768}m^2\right)\frac{e}{m}\right.$$
[25...... .5] [26..... 1] [32.................... ....2]

$$\left.-\left(\frac{9}{16}-\frac{267}{128}m\right)e'-\frac{21}{256}e'm\right]\sin(\zeta-l')$$
[33......................1] [52.......56]

(19)
$$+\left[-\left(\frac{3}{16}-\frac{3}{64}m\right)\frac{e'^2}{m}-\frac{27}{128}e'^2+\frac{9}{128}e'^2\right]\sin(\zeta-2l')$$
[32...........................3] [34.... ..1] [54......55]

(20)
$$-\frac{53}{288}\frac{e'^3}{m}\sin(\zeta-3l')$$
[32......4]

(21)
$$+\left[-\frac{3}{32}e'm+\frac{15}{32}e'm+\left(\frac{1}{4}-11\gamma^2-\frac{5}{8}e^2-\frac{3}{32}e'^2+\frac{13}{16}m-\frac{8653}{768}m^2\right)\frac{e'}{m}\right.$$
[25............2] [27.......1] [32...5]

$$\left.-\left(\frac{9}{16}+\frac{189}{128}m\right)e'+\frac{9}{256}e'm-\frac{7}{128}e'm\right]\sin(\zeta+l')$$
[35......................1] [52.......58] [55... ...55]

(22)
$$+\left[\left(\frac{3}{16}+\frac{7}{8}m\right)\frac{e'^2}{m}-\frac{27}{128}e'^2\right]\sin(\zeta+2l')$$
[32........ ....6] [36........1]

(23)
$$+\frac{53}{288}\frac{e'^3}{m}\sin(\zeta+3l')$$
[32.......7]

(24)
$$+\left[\frac{1}{4}e-\frac{7}{12}e-\left(\frac{2}{3}-\frac{80}{3}\gamma^2-\frac{5}{6}e^2-e'^2\right)\frac{e}{m^2}-\left(\frac{1}{4}-31\gamma^2-\frac{97}{16}e^2-\frac{1}{9}e'^2\right)\frac{e}{m}\right.$$
[25..18] [28...1] [32.......... ....

$$\left.-\frac{71}{36}e-\frac{11555}{2304}em-\frac{8}{9}\frac{e}{m^4}\frac{f}{n}+\frac{3}{4}e-\frac{9}{256}em\right]\sin(\zeta+l)$$
.....................................8] [37..1] [52......61]

(25)
$$-\left[\left(2+\frac{215}{16}m\right)\frac{ee'}{m}+\frac{9}{16}ee'\right]\sin(\zeta+l-l')$$
[32.......... ....9] [33......8]

(26)
$$-\frac{3}{2}\frac{ee'^2}{m}\sin(\zeta+l-2l')$$
[32....10]

(27)
$$+\left[\left(2+\frac{147}{16}m\right)\frac{ee'}{m}-\frac{9}{16}ee'\right]\sin(\zeta+l+l')$$
[32.............. ....11] [35....8]

(28)
$$+\frac{3}{2}\frac{ee'^2}{m}\sin(\zeta+l+2l')$$
[32....12]

(29)
$$+\left[\frac{3}{16}e^2+\frac{7}{12}e^2-\frac{17}{16}e^2-\left(\frac{3}{4}-\frac{545}{12}\gamma^2-\frac{9}{8}e^2-\frac{9}{8}e'^2+\frac{9}{32}m+\frac{197}{96}m^2\right)\frac{e^2}{m^2}\right.$$
[25....23] [28....18] [29....1] [32 ....13]

$$\left.+\frac{3}{4}e^2+\frac{13}{32}e^2\right]\sin(\zeta+2l)$$
[37..8] [38....1]

(30) $$-\frac{135}{32}\frac{e^3e'}{m}\sin(\zeta+2l-l')$$
[32.......14]

(31) $$+\frac{135}{32}\frac{e^3e'}{m}\sin(\zeta+2l+l')$$
[32.......15]

(32) $$-\left(\frac{8}{9}+\frac{1}{3}m\right)\frac{e^3}{m^2}\sin(\zeta+3l)$$
[32.................16]

(33) $$-\frac{625}{576}\frac{e^4}{m^2}\sin(\zeta+4l)$$
[32.....17]

(34) $$+\left[-\frac{1}{4}e+\frac{1}{4}e+\left(\frac{10}{9}-\frac{95}{9}m\right)\left(\frac{\gamma^2e}{m^2}-\frac{1}{8}\frac{e^3}{m^2}\right)+\left(\frac{2}{3}+\frac{10}{3}\gamma^2-\frac{5}{12}e^2-e'^2\right)\frac{e}{m^2}\right.$$
[25.......8] [30..1] [31 ....18] [32..........................
$$\left.+\left(\frac{1}{4}+12\gamma^2-\frac{9}{2}e^2-\frac{1}{9}e'^2\right)\frac{e}{m}+\frac{49}{288}e+\frac{1507}{1152}em+\frac{8}{9}\frac{e}{m^4}\frac{f}{n}-\frac{3}{4}e-\frac{9}{64}em\right]$$
....18] [39..1] [52....70]
$$\times\sin(\zeta-l)$$

(35) $$+\left[-\left(\frac{3}{2}+\frac{173}{16}m\right)\frac{ee'}{m}+\frac{9}{16}ee'\right]\sin(\zeta-l-l')$$
[32.........................19] [33.....18]

(36) $$-\frac{9}{8}\frac{ee'^2}{m}\sin(\zeta-l-2l')$$
[32...20]

(37) $$+\left[\left(\frac{3}{2}+\frac{157}{16}m\right)\frac{ee'}{m}+\frac{9}{16}ee'\right]\sin(\zeta-l+l')$$
[32.......................21] [35.....18]

(38) $$+\frac{9}{8}\frac{ee'^2}{m}\sin(\zeta-l+2l')$$
[32...22]

(39) $$+\left[-\frac{9}{32}e^2+\frac{1}{4}e^2+\left(\frac{5}{36}-\frac{25}{12}\gamma^2+\frac{23}{432}e^2-\frac{5}{24}e'^2-\frac{95}{72}m+\frac{403}{108}m^2\right)\frac{e^2}{m^2}\right.$$
[25........13] [30..8] [31...1]
$$+\left(\frac{1}{2}+\frac{50}{3}\gamma^2-\frac{77}{144}e^2-\frac{3}{4}e'^2-\frac{39}{32}m-\frac{1797}{1152}m^2\right)\frac{e^2}{m^2}$$
[32...23]
$$\left.+\frac{3}{4}e^2-\frac{9}{16}e^2\right]\sin(\zeta-2l)$$
[39..18] [40....1]

(40) $$-\left[\frac{39}{16}\frac{e^2e'}{m}+\frac{5}{96}\frac{e^2e'}{m}\right]\sin(\zeta-2l-l')$$
[32.......24] [31......5]

(41) $$+\left[\frac{39}{16}\frac{e^2e'}{m}+\frac{5}{96}\frac{e^2e'}{m}\right]\sin(\zeta-2l+l')$$
[32 ......25] [31......2]

(42) $$+\left[\left(\frac{17}{36}-\frac{59}{48}m\right)\frac{e^3}{m^2}+\left(\frac{5}{36}-\frac{95}{72}m\right)\frac{e^3}{m^2}\right]\sin(\zeta-3l)$$
[32.....................26] [31.....................8]

(43) $$+\left[\frac{33}{64}\frac{e^4}{m^2}+\frac{5}{32}\frac{e^4}{m^2}\right]\sin(\zeta-4l)$$
[32..... 27] [31....13]

(44) $$+\left[\frac{13}{8}\gamma^2-\frac{7}{3}\gamma^2+\gamma^2+\frac{5}{2}\frac{\gamma^2e^2}{m^2}+\left(\frac{1}{3}-\frac{19}{3}\gamma^2+\frac{33}{4}e^2-\frac{1}{2}e'^2+\frac{1}{8}m-\frac{121}{72}m^2\right)\frac{\gamma^2}{m^2}\right]$$
[25......1] [28...18] [30..8] [31....13] [32 .... 28]
$$\times\sin(\zeta+2F)$$

(45) $$+\frac{3}{8}\frac{\gamma^2e'}{m}\sin(\zeta+2F-l')$$
[32....29]

(46) $$-\frac{3}{8}\frac{\gamma^2e'}{m}\sin(\zeta+2F+l')$$
[32....30]

(47) $$+\left(1+\frac{3}{8}m\right)\frac{\gamma^2e}{m^2}\sin(\zeta+2F+l)$$
[32......... ....31]

(48) $$+\frac{17}{8}\frac{\gamma^2e^2}{m^2}\sin(\zeta+2F+2l)$$
[32......32]

(49) $$+\left[\left(\frac{10}{9}-\frac{95}{9}m\right)\frac{\gamma^2e}{m^2}+\left(4-\frac{123}{8}m\right)\frac{\gamma^2e}{m^2}\right]\sin(\zeta+2F-l)$$
[31.......................8] [32........ ....33]

(50) $$-\left[\frac{15}{4}\frac{\gamma^2e^2}{m^2}+\frac{13}{8}\frac{\gamma^2e^2}{m^2}\right]\sin(\zeta+2F-2l)$$
[31........1] [32.......34]

(51) $$-\frac{1}{4}\frac{\gamma^4}{m^2}\sin(\zeta+4F)$$
[32...35]

(52) $$+\left[\frac{1}{8}\gamma^2+\left(13-7\gamma^2-\frac{47}{4}e^2-\frac{39}{2}e'^2+\frac{27}{8}m+\frac{4135}{96}m^2\right)\frac{\gamma^2}{m^2}\right.$$
[25....28] [32.......................................1
$$\left.+3\gamma^2-3\gamma^2-\frac{3}{4}\gamma^2\right]\sin(\zeta-2F)$$
37..8] [39..18 [41...1]

(53) $$-\frac{15}{8}\frac{\gamma^2e'}{m}\sin(\zeta-2F-l')$$
[32.......5]

(54) $$+\frac{15}{8}\frac{\gamma^2e'}{m}\sin(\zeta-2F+l')$$
[32.......2]

(55) $$-\left(\frac{4}{3}-\frac{225}{8}m\right)\frac{\gamma^2e}{m^2}\sin(\zeta-2F+l)$$
[32....................18]

(56) $$+\left[+\frac{83}{12}\frac{\gamma^2e^2}{m^2}+\frac{23}{8}\frac{\gamma^2e^2}{m^2}\right]\sin(\zeta-2F+2l)$$
[32............23] [42.......7

(57) $$+\left(\frac{79}{3}+\frac{239}{8}m\right)\frac{\gamma^2 e}{m^2}\sin(\zeta-2F-l)$$
[32 .... 8]

(58) $$+\left[-\frac{5}{72}\frac{\gamma^2 e^2}{m^2}+\frac{533}{12}\frac{\gamma^2 e^2}{m^2}\right]\sin(\zeta-2F-2l)$$
[31 .... 28] [32 .... 13]

(59) $$-\frac{79}{12}\frac{\gamma^4}{m^2}\sin(\zeta-4F)$$
[32 .... 28]

(60) $$+\left[\frac{3}{32}m+\frac{25}{128}m^2+\left(\frac{1}{4}\gamma^2-\frac{45}{16}e^2\right)\frac{1}{m}-\frac{11}{24}+\frac{2743}{96}\gamma^2-\frac{1421}{128}e^2+\frac{11}{6}e'^2\right.$$
[25 .... 55] [32 ....

$$\left.-\frac{505}{288}m-\frac{5333}{864}m^2-\frac{23}{64}m^2-\frac{3}{32}m+\frac{33}{128}m^2\right]\sin(\zeta+2D)$$
.... 36] [43 .... 1] [45 .... 1]

(61) $$+\left[\frac{7}{32}e'm+\left(\frac{7}{12}\gamma^2-\frac{105}{16}e^2-\frac{77}{48}m-\frac{4129}{384}m^2\right)\frac{e'}{m}-\frac{7}{32}e'm\right]\sin(\zeta+2D-l')$$
[25 .... 56] [32 .... 37] [46 .... 1]

(62) $$-\frac{187}{48}e'^2\sin(\zeta+2D-2l')$$
[32 ..... 38]

(63) $$+\left[-\frac{3}{32}e'm-\left(\frac{1}{4}\gamma^2-\frac{45}{16}e^2-\frac{11}{48}m-\frac{2353}{1152}m^2\right)\frac{e'}{m}+\frac{3}{32}e'm\right]\sin(\zeta+2D+l')$$
[25 .... 58] [32 .... 39] [47 .... 1]

(64) $$+\left[\frac{3}{32}em+\frac{7}{32}em+\left(\frac{3}{4}\gamma^2-5e^2-\frac{7}{6}m-\frac{637}{144}m^2\right)\frac{e}{m}-\frac{3}{32}em-\frac{7}{32}em\right]$$
[25 ...... 61] [28 ..... 55] [32 .... 40] [45 ...... 8] [48 ...... 1]

$$\times\sin(\zeta+2D+l)$$

(65) $$-\frac{49}{12}ee'\sin(\zeta+2D+l-l')$$
[32 .... 41]

(66) $$+\frac{7}{12}ee'\sin(\zeta+2D+l+l')$$
[32 .... 42]

(67) $$-\frac{425}{192}e^2\sin(\zeta+2D+2l)$$
[32 .... 43]

(68) $$+\left[\frac{3}{8}em-\frac{3}{32}em+\left(\frac{5}{12}\gamma^2-\frac{5}{96}e^2\right)\frac{e}{m}-\left(\frac{5}{4}-53\gamma^2-\frac{55}{32}e^2-5e'^2\right.\right.$$
[25 ... 70] [30 ..... 55] [31 .... 61] [32 ....

$$\left.\left.+\frac{527}{96}m+\frac{57299}{2304}m^2\right)\frac{e}{m}-\frac{35}{32}em+\frac{3}{32}em+\frac{3}{2}em\right]\sin(\zeta+2D-l)$$
.... 44] [44 ...... 1] [45 .... 18] [49 ...... 1]

(69) $$-\left(\frac{35}{12}+\frac{599}{32}m\right)\frac{ee'}{m}\sin(\zeta+2D-l-l')$$
[32 .... 45]

(70) $$-\frac{85}{16}\frac{ee'^2}{m}\sin(\zeta + 2D - l - 2l')$$
[32... .46]

(71) $$+\left(\frac{5}{4}+\frac{241}{96}m\right)\frac{ee'}{m}\sin(\zeta + 2D - l + l')$$
[32....................47]

(72) $$+\frac{15}{16}\frac{ee'^2}{m}\sin(\zeta + 2D - l + 2l')$$
[32.....48]

(73) $$+\left[-\left(\frac{5}{96}-\frac{445}{1152}m\right)\frac{e^2}{m}+\left(\frac{5}{32}+\frac{25}{12}m\right)\frac{e^2}{m}-\frac{15}{16}e^2\right]\sin(\zeta + 2D - 2l)$$
[31............................55] [32.................... 49] [50... 1]

(74) $$+\left[-\frac{35}{288}\frac{e^2e'}{m}+\frac{35}{96}\frac{e^2e'}{m}\right]\sin(\zeta + 2D - 2l - l')$$
[31............56] [32.....50]

(75) $$+\left[\frac{5}{96}\frac{e^2e'}{m}-\frac{15}{96}\frac{e^2e'}{m}\right]\sin(\zeta + 2D - 2l + l')$$
[31.......58] [32.....51]

(76) $$+\left[-\frac{5}{24}\frac{e^3}{m}-\frac{5}{8}\frac{e^3}{m}\right]\sin(\zeta + 2D - 3l)$$
[31.........70] [32..52]

(77) $$+\frac{11}{16}\gamma^2\sin(\zeta + 2D + 2F)$$
[32....53]

(78) $$+\frac{15}{8}\frac{\gamma^2 e}{m}\sin(\zeta + 2D + 2F - l)$$
[32.....54]

(79) $$+\left[\left(\frac{17}{4}+\frac{1199}{96}m\right)\frac{\gamma^2}{m}+\frac{9}{16}\gamma^2\right]\sin(\zeta + 2D - 2F)$$
[32.........................55] [51......1]

(80) $$+\frac{119}{12}\frac{\gamma^2 e'}{m}\sin(\zeta + 2D - 2F - l')$$
[32.......56]

(81) $$-\frac{17}{4}\frac{\gamma^2 e'}{m}\sin(\zeta + 2D - 2F + l')$$
[32......58]

(82) $$+\frac{85}{8}\frac{\gamma^2 e}{m}\sin(\zeta + 2D - 2F + l)$$
[32......61]

(83) $$-\frac{3}{8}\frac{\gamma^2 e}{m}\sin(\zeta + 2D - 2F - l)$$
[32....70]

(84) $$+\left[-\frac{11}{64}m^2+\left(\frac{1}{4}+\frac{21}{4}\gamma^2+\frac{9}{16}e^2-e'^2\right)\frac{1}{m}+\frac{59}{96}+\frac{649}{96}\gamma^2+\frac{33}{128}e^2-\frac{333}{64}e'^2\right.$$
[25..........36] [32...

$$+\frac{5255}{2304}m+\frac{205927}{27648}m^2+\frac{1}{3}\frac{1}{m^3}\frac{f}{n}+\frac{3}{32}-\frac{15}{8}\gamma^2+\frac{9}{32}e^2-\frac{15}{64}e'^2-\frac{11}{256}m$$
....55] [52... ....

$$\left.-\frac{127}{3072}m^2-\frac{3}{8}m^2\right]\sin(\zeta - 2D)$$
..........1] [59....1]

(85) $$+\left[-\left(\frac{1}{4}+\frac{21}{4}\gamma^2+\frac{9}{16}e^2-\frac{13}{32}e'^2+\frac{31}{24}m+\frac{7049}{2304}m^2\right)\frac{e'}{m}+\frac{27}{128}e'm-\frac{9}{256}e'm\right.$$
[32............58] [33.......55] [52........5]

$$\left.-\frac{77}{128}e'm\right]\sin(\zeta-2D-l')$$
[53........1]

(86) $$+\left[-\left(\frac{3}{16}+\frac{7}{32}m\right)\frac{e'^2}{m}-\left(\frac{3}{16}+\frac{5}{16}m\right)\frac{e'^2}{m}\right]\sin(\zeta-2D-2l')$$
[32..........59] [54..........1]

(87) $$-\frac{1}{96}\frac{e'^3}{m}\sin(\zeta-2D-3l')$$
[32....60]

(88) $$+\left[\left(\frac{7}{12}+\frac{49}{4}\gamma^2+\frac{21}{16}e^2-\frac{69}{32}e'^2+\frac{23}{8}m+\frac{27661}{2304}m^2\right)\frac{e'}{m}+\frac{27}{128}e'm+\frac{9}{256}e'm\right.$$
[32............56] [35........55] [52........2]

$$\left.+\left(\frac{7}{48}+\frac{23}{384}m\right)e'\right]\sin(\zeta-2D+l')$$
[55..........1]

(89) $$+\left[\left(\frac{17}{16}+\frac{1441}{192}m\right)\frac{e'^2}{m}+\frac{51}{256}e'^2\right]\sin(\zeta-2D+2l')$$
[32..........57] [56.......1]

(90) $$+\left[-\frac{15}{32}em+\left(1-\frac{17}{4}\gamma^2-\frac{1}{2}e^2-4e'^2+\frac{19}{4}m+\frac{1153}{48}m^2\right)\frac{e}{m}-\frac{9}{32}em\right.$$
[25 ....44] [32............70] [37.....55]

$$\left.-\left(\frac{3}{32}-\frac{11}{256}m\right)e+\frac{3}{4}em-\frac{45}{32}em\right]\sin(\zeta-2D+l)$$
[52..........18] [57....1] [60......1]

(91) $$-\left(1+\frac{7}{8}m\right)\frac{ee'}{m}\sin(\zeta-2D+l-l')$$
[32..........73]

(92) $$+\left[-\frac{3}{4}\frac{ee'^2}{m}+\frac{3}{16}\frac{ee'^2}{m}\right]\sin(\zeta-2D+l-2l')$$
[32..........74] [54.....18]

(93) $$+\left[\left(\frac{7}{3}+\frac{121}{8}m\right)\frac{ee'}{m}-\frac{7}{48}ee'\right]\sin(\zeta-2D+l+l')$$
[32..........71] [55....18]

(94) $$+\frac{17}{4}\frac{ee'^2}{m}\sin(\zeta-2D+l+2l')$$
[32.....72]

(95) $$+\left[\left(\frac{49}{32}+\frac{6955}{768}m\right)\frac{e^2}{m}-\frac{9}{128}e^2+\frac{195}{256}e^2\right]\sin(\zeta-2D+2l)$$
[32..........75] [52.....23] [61... ..1]

(96) $$-\frac{49}{32}\frac{e^2e'}{m}\sin(\zeta-2D+2l-l')$$
[32. ...77]

(97) $$+\frac{343}{96}\frac{e^2e'}{m}\sin(\zeta-2D+2l+l')$$
[32.......76]

(98) $$+\frac{67}{24}\frac{e^3}{m}\sin(\zeta-2D+3l)$$
[32...78]

(99) $$+\Big[-\Big(\frac{25}{12}\gamma^2-\frac{25}{96}e^2\Big)\frac{e}{m}+\Big(\frac{1}{4}+\frac{15}{2}\gamma^2+\frac{41}{32}e^2-e'^2+\frac{55}{96}m+\frac{4339}{2304}m^2\Big)\frac{e}{m}$$
[31... ....44] [32........................... . ...61]

$$+\frac{9}{32}em+\Big(\frac{3}{32}-\frac{11}{256}m\Big)e-\frac{9}{32}em\Big]\sin(\zeta-2D-l)$$
[39.....55] [52....................8] [58.......1]

(100) $$-\Big(\frac{1}{4}+\frac{185}{96}m\Big)\frac{ee'}{m}\sin(\zeta-2D-l-l')$$
[32... ....64]

(101) $$+\Big[-\frac{3}{16}\frac{ee'^2}{m}-\frac{3}{16}\frac{ee'^2}{m}\Big]\sin(\zeta-2D-l-2l')$$
[32...........65] [54......8]

(102) $$+\Big[\Big(\frac{7}{12}+\frac{325}{96}m\Big)\frac{ee'}{m}+\frac{7}{48}ee'\Big]\sin(\zeta-2D-l+l')$$
[32......... ....62] [55......8]

(103) $$+\frac{17}{16}\frac{ee'^2}{m}\sin(\zeta-2D-l+2l')$$
[32.....63]

(104) $$+\Big[\frac{55}{576}e^2+\Big(\frac{9}{32}+\frac{229}{256}m\Big)\frac{e^2}{m}+\frac{27}{256}e^2\Big]\sin(\zeta-2D-2l)$$
[31......36] [32... ....66] [52.....13]

(105) $$-\frac{9}{32}\frac{e^2e'}{m}\sin(\zeta-2D-2l-l')$$
[32.....68]

(106) $$+\frac{21}{32}\frac{e^2e'}{m}\sin(\zeta-2D-2l+l')$$
[32.....67]

(107) $$+\frac{1}{3}\frac{e^3}{m}\sin(\zeta-2D-3l)$$
[32..69]

(108) $$+\Big[\Big(\frac{15}{8}-\frac{137}{64}m\Big)\frac{\gamma^2}{m}-\frac{81}{64}\gamma^2\Big]\sin(\zeta-2D+2F)$$
[32......................79] [52......1]

(109) $$-\frac{15}{8}\frac{\gamma^2e'}{m}\sin(\zeta-2D+2F-l')$$
[32......81]

(110) $$+\frac{35}{8}\frac{\gamma^2e'}{m}\sin(\zeta-2D+2F+l')$$
[32......80]

(111) $$+\frac{21}{4}\frac{\gamma^2e}{m}\sin(\zeta-2D+2F+l)$$
[32......83]

(112) $$-\Big[\frac{5}{3}\frac{\gamma^2e}{m}+\frac{33}{8}\frac{\gamma^2e}{m}\Big]\sin(\zeta-2D+2F-l)$$
[31......70] [32.....82]

(113) $$-\left[\left(\frac{1}{8}-\frac{1623}{64}m\right)\frac{\gamma^2}{m}+\frac{3}{64}\gamma^2\right]\sin(\zeta-2D-2F)$$
[32..... ... .36] [52.....28]

(114) $$+\frac{1}{8}\frac{\gamma^2 e'}{m}\sin(\zeta-2D-2F-l')$$
[32....39]

(115) $$-\frac{7}{24}\frac{\gamma^2 e'}{m}\sin(\zeta-2D-2F+l')$$
[32. ....37]

(116) $$-\frac{353}{8}\frac{\gamma^2 e}{m}\sin(\zeta-2D-2F+l)$$
[32......44]

(117) $$-\frac{3}{8}\frac{\gamma^2 e}{m}\sin(\zeta-2D-2F-l)$$
[32....40]

(118) $$-\frac{161}{384}m^2\sin(\zeta+4D)$$
[32....84]

(119) $$-\frac{35}{16}em\sin(\zeta+4D-l)$$
[32....85]

(120) $$-\frac{675}{256}e^2\sin(\zeta+4D-2l)$$
[32....86]

(121) $$+\frac{3}{64}\gamma^2\sin(\zeta+4D-2F)$$
[32....87]

(122) $$+\left[-\frac{3}{32}\gamma^2+\frac{135}{128}e^2+\frac{11}{64}m+\frac{447}{512}m^2+\frac{33}{512}m^3\right]\sin(\zeta-4D)$$
[32...87] [52......36]

(123) $$-\frac{33}{128}e'm\sin(\zeta-4D-l')$$
[32.......89]

(124) $$+\frac{385}{384}e'm\sin(\zeta-4D+l')$$
[32.......88]

(125) $$+\left[\left(\frac{15}{32}+\frac{757}{256}m\right)e+\frac{45}{256}em\right]\sin(\zeta-4D+l)$$
[32.......................91] [52.......44]

(126) $$-\frac{15}{16}ee'\sin(\zeta-4D+l-l')$$
[32....93]

(127) $$+\frac{35}{16}ee'\sin(\zeta-4D+l+l')$$
[32....92]

(128) $$+\frac{195}{256}e^2\sin(\zeta-4D+2l)$$
[32....94]

(129) $$+\frac{7}{16}em\sin(\zeta-4D-l)$$
[3a.....90]

(130) $$+\frac{45}{64}\gamma^2\sin(\zeta-4D+2F)$$
[3a....95]

(131) $$+\left(\frac{5}{8}+\frac{709}{192}m\right)\frac{1}{m}\frac{a}{a'}\sin(\zeta+D)$$
[3a......................96]

(132) $$-\frac{5}{8}\frac{e'}{m}\frac{a}{a'}\sin(\zeta+D-l')$$
[3a... 97]

(133) $$+\left[-\left(\frac{5}{6}-\frac{55}{16}m\right)\frac{e'}{m^2}+\left(\frac{100}{117}\gamma^2+\frac{25}{117}e^2\right)\frac{e'}{m^3}\right]\frac{a}{a'}\sin(\zeta+D+l')$$
[3a........................98] [6a... ... .8]

(134) $$+\frac{45}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta+D+l)$$
[3a... ..99]

(135) $$-\frac{15}{8}\frac{ee'}{m^3}\frac{a}{a'}\sin(\zeta+D+l+l')$$
[3a......100]

(136) $$-\frac{15}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta+D-l)$$
[3a.....101]

(137) $$+\left[\frac{5}{24}\frac{ee'}{m^2}+\left(\frac{25}{117}+\frac{400}{1521}\frac{\gamma^2}{m}+\frac{250}{4563}\frac{e^2}{m}-\frac{8855}{9126}m\right)\frac{ee'}{m^3}\right]\frac{a}{a'}\sin(\zeta+D-l+l')$$
[3a.....102] [6a...1]

(138) $$-\frac{25}{117}\frac{e^2e'}{m^3}\frac{a}{a'}\sin(\zeta+D-2l+l')$$
[6a..........18]

(139) $$+\frac{100}{117}\frac{\gamma^2e'}{m^3}\frac{a}{a'}\sin(\zeta+D-2F+l')$$
[6a..........18]

(140) $$+\frac{200}{1521}\frac{\gamma^2ee'}{m^4}\frac{a}{a'}\sin(\zeta+D-2F-l+l')$$
[6a..............1]

(141) $$-\left(\frac{5}{8}+\frac{19}{8}m\right)\frac{1}{m}\frac{a}{a'}\sin(\zeta-D)$$
[3a.....................103]

(142) $$+\left(\frac{5}{6}-\frac{55}{16}m\right)\frac{e'}{m^2}\frac{a}{a'}\sin(\zeta-D-l')$$
[3a.....................105]

(143) $$+\frac{5}{16}\frac{e'}{m}\frac{a}{a'}\sin(\zeta-D+l')$$
[3a.....104]

(144) $$-\frac{15}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta-D+l)$$
[3a.....108]

(145) $$+\left[\frac{25}{24}\frac{ee'}{m^2}-\frac{5}{12}\frac{ee'}{m^2}\right]\frac{a}{a'}\sin(\zeta-D+l-l')$$
[32.....109] [64......1]

(146) $$-\frac{65}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta-D-l)$$
[32 ....106]

(147) $$+\left[\frac{55}{72}\frac{ee'}{m^2}+\frac{5}{18}\frac{ee'}{m^2}\right]\frac{a}{a'}\sin(\zeta-D-l-l')$$
[32.....107] [63.....1

(148 $$-\frac{25}{312}\frac{ee'}{m^2}\frac{a}{a'}\sin(\zeta-D-l+l')$$
[62........55]

(149) $$-\frac{5}{32}\frac{a}{a'}\sin(\zeta+3D)$$
[32..110]

(150) $$-\frac{95}{192}\frac{a}{a'}\sin(\zeta-3D)$$
[32....111]

(151) $$+\frac{5}{16}\frac{e'}{m}\frac{a}{a'}\sin(\zeta-3D-l')$$
[32.....112]

(152) $$-\frac{25}{48}\frac{e}{m}\frac{a}{a'}\sin(\zeta-3D+l)\Big\}$$
[32.....113]

(153) $$+\frac{\beta_3}{a^3}\Big\{\left[\frac{3}{4}\gamma-\frac{7}{6}\gamma+\frac{1}{2}\gamma-\frac{1}{3}\frac{\gamma^3}{m^2}-\frac{1}{8}\frac{\gamma^3}{m}\right]\sin(2\zeta+F)$$
[65...........1] [68..18] [73...8] [79.... 28

(154) $$+\left[\frac{5}{2}\gamma e+\frac{7}{8}\gamma e-\frac{17}{4}\gamma e+\frac{9}{8}\gamma e-\frac{\gamma^3 e}{m^2}\right]\sin(2\zeta+F+l)$$
[65.....8] [68....1] [71....18] [73..13] [79..31]

(155) $$+\left[\frac{3}{2}\gamma e-\frac{7}{4}\gamma e-\frac{11}{8}\gamma e-\frac{10}{3}\frac{\gamma^3 e}{m^2}-\frac{5}{12}\frac{\gamma e^2}{m^2}-\frac{1}{12}\gamma e+\frac{35}{24}\gamma e-4\frac{\gamma^3 e}{m^2}\right]$$
[65.....18] [68..23] [73.....1] [76........ ....8] [78......1] [79....33
$$\times\sin(2\zeta+F-l)$$

(156) $$-\left(\frac{5}{12}-\frac{85}{16}m\right)\frac{\gamma e^2}{m^2}\sin(2\zeta+F-2l)$$
[76.....................1]

(157) $$+\frac{5}{12}\frac{\gamma e^2}{m^2}\sin(2\zeta+F-3l)$$
[76.....18]

(158) $$+\left[\frac{5}{4}\gamma-\frac{7}{6}\gamma+\frac{1}{2}\gamma+\left(\frac{2}{3}-\frac{17}{3}\gamma^2-\frac{2}{3}e^2-e'^2\right)\frac{\gamma}{m^2}+\left(\frac{1}{4}-\frac{23}{8}\gamma^2-6e^2-\frac{1}{9}e'^2\right)\frac{\gamma}{m}\right.$$
[65....1] [68..8] [73..18] [79...................................... ....
$$\left.+\frac{20}{9}\gamma+\frac{13319}{2304}\gamma m+\frac{8}{9}\frac{\gamma}{m^4}\frac{f}{n}+\frac{9}{128}\gamma m\right]\sin(2\zeta-F)$$
....1] [91........55]

(159) $$+\left[\left(\frac{1}{4}+\frac{3}{16}m\right)\frac{\gamma e'}{m}+\frac{9}{4}\gamma e'\right]\sin(2\zeta-F-l')$$
[79............ ....2] [80.....1]

$$+\frac{3}{16}\frac{\gamma e'^2}{m}\sin(2\zeta - F - 2l') \tag{160}$$
[79.......3]

$$+\left[-\left(\frac{1}{4}+\frac{13}{16}m\right)\frac{\gamma e'}{m}+\frac{9}{4}\gamma e'\right]\sin(2\zeta - F + l') \tag{161}$$
[79..........................5] [81.....1]

$$-\frac{3}{16}\frac{\gamma e'^2}{m}\sin(2\zeta - F + 2l') \tag{162}$$
[79......6]

$$+\left[\frac{1}{2}\gamma e+\frac{35}{8}\gamma e-\frac{17}{4}\gamma e+\frac{3}{4}\gamma e+\left(\frac{2}{3}-\frac{37}{3}\gamma^2-\frac{5}{6}e^2-e'^2+\frac{1}{4}m+\frac{37}{18}m^2\right)\frac{\gamma e}{m^2}-\frac{17}{8}\gamma e\right]\sin(2\zeta - F + l) \tag{163}$$
[65....18] [68.....1] [71....8] [73..23] [79..........................18] [82.....1]

$$+2\frac{\gamma ee'}{m}\sin(2\zeta - F + l - l') \tag{164}$$
[79......9]

$$-2\frac{\gamma ee'}{m}\sin(2\zeta - F + l + l') \tag{165}$$
[79.....11]

$$+\left[\frac{3}{4}\frac{\gamma e^2}{m^2}+\frac{9}{32}\frac{\gamma e^2}{m}\right]\sin(2\zeta - F + 2l) \tag{166}$$
[79......... ....13]

$$+\frac{8}{9}\frac{\gamma e^3}{m^2}\sin(2\zeta - F + 3l) \tag{167}$$
[79...16]

$$+\left[\frac{7}{2}\gamma e-\frac{21}{8}\gamma e-\frac{7}{8}\gamma e-\frac{10}{3}\frac{\gamma^3 e}{m^2}+\frac{5}{12}\frac{\gamma e^3}{m^2}-\frac{1}{12}\gamma e-\left(\frac{2}{3}+\frac{13}{3}\gamma^2-\frac{5}{12}e^2-e'^2+\frac{1}{4}m+\frac{73}{288}m^2\right)\frac{\gamma e}{m^2}+\frac{3}{2}\gamma e\right]\sin(2\zeta - F - l) \tag{168}$$
[65......8] [68....13] [73...1] [76.......................18] [79..........................18] [83. ..1]

$$+\frac{3}{2}\frac{\gamma ee'}{m}\sin(2\zeta - F - l - l') \tag{169}$$
[79.....19]

$$-\frac{3}{2}\frac{\gamma ee'}{m}\sin(2\zeta - F - l + l') \tag{170}$$
[79.....21]

$$+\left[-\left(\frac{5}{12}-\frac{85}{16}m\right)\frac{\gamma e^2}{m^2}-\left(\frac{1}{2}-\frac{39}{32}m\right)\frac{\gamma e^2}{m^2}\right]\sin(2\zeta - F - 2l) \tag{171}$$
[76............................1] [79.....................23]

$$+\left[-\frac{5}{12}\frac{\gamma e^3}{m^2}-\frac{17}{36}\frac{\gamma e^3}{m^2}\right]\sin(2\zeta - F - 3l) \tag{172}$$
[76............8] [79.....26]

$$-\left(\frac{19}{3}+\frac{17}{8}m\right)\frac{\gamma^3}{m^2}\sin(2\zeta - 3F) \tag{173}$$
[79.....................1]

(174) $$+\frac{4}{3}\frac{\gamma^3 e}{m^3}\sin(2\zeta - 3F + l)$$
[79...18]

(175) $$-13\frac{\gamma^3 e}{m^3}\sin(2\zeta - 3F - l)$$
[79......8]

(176) $$+\left[\frac{35}{16}\gamma m - \frac{35}{16}\gamma m\right]\sin(2\zeta + 2D + F)$$
[68........44] [86.......8]

(177) $$+\left[\frac{21}{32}\gamma m + \frac{7}{4}\gamma m + \frac{3}{16}\gamma m + \left(-\frac{1}{4}\gamma^2 + \frac{45}{16}e^2 + \frac{11}{24}m + \frac{1043}{576}m^2\right)\frac{\gamma}{m} - \frac{35}{16}\gamma m\right.$$
[65.......55] [68....70] [73......61] [79..36] [86. ...18]

$$\left.-\frac{3}{8}\gamma m\right]\sin(2\zeta + 2D - F)$$
[90.....1]

(178) $$+\frac{77}{48}\gamma e'\sin(2\zeta + 2D - F - l')$$
[79.....37]

(179) $$-\frac{11}{48}\gamma e'\sin(2\zeta + 2D - F + l')$$
[79... .39]

(180) $$+\frac{7}{6}\gamma e\sin(2\zeta + 2D - F + l)$$
[79..40]

(181) $$+\left(\frac{5}{4} + \frac{527}{96}m\right)\frac{\gamma e}{m}\sin(2\zeta + 2D - F - l)$$
[79....................44]

(182) $$+\frac{35}{12}\frac{\gamma e e'}{m}\sin(2\zeta + 2D - F - l - l')$$
[79......45]

(183) $$-\frac{5}{4}\frac{\gamma e e'}{m}\sin(2\zeta + 2D - F - l + l')$$
[79.. ...47]

(184) $$+\left[\frac{5}{32}\frac{\gamma e^2}{m} - \frac{5}{32}\frac{\gamma e^2}{m}\right]\sin(2\zeta + 2D - F - 2l)$$
[76.......55] [79.....49]

(185) $$-\frac{9}{4}\frac{\gamma^3}{m}\sin(2\zeta + 2D - 3F)$$
[79...55]

(186) $$+\left[-\frac{9}{32}\gamma m - \frac{7}{16}\gamma m - \frac{3}{4}\gamma m - \frac{15}{8}\frac{\gamma^3}{m} - \left(\frac{3}{16} - \frac{1}{64}m\right)\gamma + \frac{15}{16}\gamma m + \frac{3}{4}\gamma m\right]$$
[65...........55] [68.. ...61] [73....70] [79....79] [91.....................1] [93......18] [98......1]

$$\times\sin(2\zeta - 2D + F)$$

(187) $$+\frac{3}{8}\gamma e'\sin(2\zeta - 2D + F - l')$$
[103...1]

(188) $$-\frac{7}{24}\gamma e'\sin(2\zeta - 2D + F + l')$$
[98......1]

(189) $$-\frac{3}{16}\gamma e \sin(2\zeta - 2D + F + l)$$
[91.....8]

(190) $$+\frac{3}{16}\gamma e \sin(2\zeta - 2D + F - l)$$
[91....18]

(191) $$+\frac{5}{32}\frac{\gamma e^2}{m}\sin(2\zeta - 2D + F - 2l)$$
[76....55]

(192) $$+\left[-\frac{15}{16}\gamma m - \left(\frac{1}{4} + \frac{29}{8}\gamma^2 + \frac{9}{16}e^2 - e'^2 + \frac{59}{96}m + \frac{5327}{2304}m^2\right)\frac{\gamma}{m} - \left(\frac{3}{16} - \frac{1}{64}m\right)\gamma + \frac{15}{16}\gamma m\right]\sin(2\zeta - 2D - F)$$
[73..........44] [79..55] [91....................1] [93.......8]

(193) $$+\left(\frac{1}{4} + \frac{31}{24}m\right)\frac{\gamma e'}{m}\sin(2\zeta - 2D - F - l')$$
[79....................58]

(194) $$+\frac{3}{16}\frac{\gamma e'^2}{m}\sin(2\zeta - 2D - F - 2l')$$
[79......59]

(195) $$-\left[\left(\frac{7}{12} + \frac{23}{8}m\right)\frac{\gamma e'}{m} + \frac{7}{24}\gamma e'\right]\sin(2\zeta - 2D - F + l')$$
[79.... ....56] [92... ...1]

(196) $$-\frac{17}{16}\frac{\gamma e'^2}{m}\sin(2\zeta - 2D - F + 2l')$$
[79......57]

(197) $$+\left[-\left(1 + \frac{19}{4}m\right)\frac{\gamma e}{m} + \frac{3}{16}\gamma e\right]\sin(2\zeta - 2D - F + l)$$
[79.........................70] [91....18]

(198) $$+\frac{\gamma e e'}{m}\sin(2\zeta - 2D - F + l - l')$$
[79..73]

(199) $$-\frac{7}{3}\frac{\gamma e e'}{m}\sin(2\zeta - 2D - F + l + l')$$
[79... 71]

(200) $$-\frac{49}{32}\frac{\gamma e^2}{m}\sin(2\zeta - 2D - F + 2l)$$
[79.....75]

(201) $$-\left[\left(\frac{1}{4} + \frac{55}{96}m\right)\frac{\gamma e}{m} - \frac{3}{16}\gamma e\right]\sin(2\zeta - 2D - F - l)$$
[79.....................61] [91.. ...8]

(202) $$+\frac{1}{4}\frac{\gamma e e'}{m}\sin(2\zeta - 2D - F - l - l')$$
[79.....64]

(203) $$-\frac{7}{12}\frac{\gamma e e'}{m}\sin(2\zeta - 2D - F - l + l')$$
[79......62]

(204) $$-\frac{9}{32}\frac{\gamma e^2}{m}\sin(2\zeta-2D-F-2l)$$
[79.....66]

(205) $$+\frac{1}{8}\frac{\gamma^3}{m}\sin(2\zeta-2D-3F)$$
[79..36]

(206) $$+\left[\frac{9}{128}\gamma m-\frac{9}{256}\gamma m\right]\sin(2\zeta-4D+F)$$
[91.........55] [100.......1]

(207) $$-\frac{11}{64}\gamma m\sin(2\zeta-4D-F)$$
[79.....87]

(208) $$-\frac{15}{32}\gamma e\sin(2\zeta-4D-F+l)$$
[79....91]

(209) $$-\frac{5}{8}\frac{\gamma}{m}\frac{a}{a'}\sin(2\zeta+D-F)$$
[79.....96]

(210) $$+\frac{5}{6}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(2\zeta+D-F+l')$$
[79......98]

(211) $$+\frac{5}{8}\frac{\gamma}{m}\frac{a}{a'}\sin(2\zeta-D-F)$$
[79....103]

(212) $$-\frac{5}{6}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(2\zeta-D-F-l')\Big\}.$$
[79.....105]

$$\frac{a}{r}= \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

(1) $$+\frac{\beta_1}{a^3}\left[1-\frac{4}{3}\right]$$
[1.......2] [108..1]

(2) $$+\frac{\beta_2}{a^3}\Big\{\left[\frac{5}{9}\frac{\gamma e}{m^2}+\frac{5}{3}\frac{\gamma e}{m^2}\right]\cos(\zeta+F-l)$$
[31............2] [32. .3]

(3) $$+\frac{20}{3}\frac{\gamma e}{m^2}\cos(\zeta-F+l)$$
[32.....2]

(4) $$-\frac{20}{3}\frac{\gamma e}{m^2}\cos(\zeta-F-l)\Big\}$$
[32.....2]

(5) $$+\frac{\beta_3}{a^3}\left[-1+\frac{7}{12}+\frac{1}{4}\right]\cos 2\zeta.$$
[65.........1] [68..2] [73..2]

CHAPTER IV.

REDUCED EXPRESSIONS FOR THE PERTURBATIONS OF THE CO-ORDINATES OF THE MOON PRODUCED BY THE FIGURE OF THE EARTH.

The expressions of the preceding chapter, being reduced, lead to the following:

$V = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$

(1) $\quad + \frac{\beta_1}{a^2} \Big\{ - 6e'm \sin l'$

(2) $\quad + \left[\frac{5}{3} \frac{\gamma^2 e^2}{m^2} - \frac{17}{12} e^2 \right] \sin 2l$

(3) $\quad + \left[\frac{25}{3} \frac{\gamma^2 e^2}{m^2} + \frac{1}{3} \gamma^2 \right] \sin 2F$

(4) $\quad + \left[\frac{20}{3} \frac{\gamma^2 e}{m^2} - \frac{105}{2} \frac{\gamma^2 e}{m} \right] \sin (2F - l)$

(5) $\quad - \frac{55}{3} \frac{\gamma^2 e^2}{m^2} \sin (2F - 2l)$

(6) $\quad + \left[\frac{3}{4} \gamma^2 + \frac{75}{16} e^2 - \frac{2}{3} m^2 \right] \sin 2D$

(7) $\quad + \left[\frac{15}{4} e + 17em \right] \sin (2D - l)$

(8) $\quad + \frac{35}{6} ee' \sin (2D - l - l')$

(9) $\quad - \frac{15}{2} ee' \sin (2D - l + l')$

(10) $\quad + \frac{15}{4} \frac{ee'^2}{m} \sin (2D - l + 2l')$

(11) $\quad - \frac{15}{4} e^2 \sin (2D - 2l)$

(12) $\quad + 3\gamma^2 \sin (2D - 2F)$

(13) $\quad + \frac{25}{2} \frac{\gamma^2 e}{m} \sin (2D - 2F + l)$

(14) $$-\frac{15}{8}\frac{a}{a'}\sin D$$

(15) $$-\left[\frac{10}{3}\frac{e'}{m^2}-\frac{185}{4}\frac{e'}{m}\right]\frac{a}{a'}\sin(D+l')$$

(16) $$-\frac{25}{6}\frac{ee'}{m^2}\frac{a}{a'}\sin(D+l+l')$$

(17) $$-\frac{25}{4}\frac{e'}{m}\frac{a}{a'}\sin(D-l')$$

(18) $$+\frac{25}{2}\frac{ee'}{m^2}\frac{a}{a'}\sin(D-l+l')\Big\}$$

(19) $$+\frac{\beta_2}{a^2}\Big\{\Big[\Big(\frac{2}{3}\gamma-\frac{37}{3}\gamma^3+\frac{26}{9}\gamma e^2-\gamma e'^2+\frac{8}{9}\frac{\gamma}{m^2}\frac{f}{n}\Big)\frac{1}{m^2}+\Big(\frac{1}{4}\gamma-\frac{33}{8}\gamma^3$$
$$-\frac{292}{9}\gamma e^2-\frac{1}{9}\gamma e'^2\Big)\frac{1}{m}+\frac{13}{18}\gamma+\frac{4439}{576}\gamma m\Big]\sin(\zeta+F)$$

(20) $$+\left[\frac{1}{2}\frac{\gamma e'}{m}-\frac{29}{16}\gamma e'\right]\sin(\zeta+F-l')$$

(21) $$+\frac{3}{8}\frac{\gamma e'^2}{m}\sin(\zeta+F-2l')$$

(22) $$-\left[\frac{1}{2}\frac{\gamma e'}{m}+\frac{61}{16}\gamma e'\right]\sin(\zeta+F+l')$$

(23) $$-\frac{3}{8}\frac{\gamma e'^2}{m}\sin(\zeta+F+2l')$$

(24) $$+\left[\left(\frac{4}{3}\gamma e-\frac{114}{3}\gamma^3 e+\frac{49}{18}\gamma e^3-2\gamma ee'^2\right)\frac{1}{m^2}+\frac{1}{2}\frac{\gamma e}{m}+\frac{23}{18}\gamma e\right]\sin(\zeta+F+l)$$

(25) $$+\frac{9}{2}\frac{\gamma ee'}{m}\sin(\zeta+F+l-l')$$

(26) $$-\frac{9}{2}\frac{\gamma ee'}{m}\sin(\zeta+F+l+l')$$

(27) $$+\left[\frac{13}{6}\frac{\gamma e^2}{m^2}+\frac{13}{16}\frac{\gamma e^2}{m}\right]\sin(\zeta+F+2l)$$

(28) $$+\frac{59}{18}\frac{\gamma e^3}{m^2}\sin(\zeta+F+3l)$$

(29) $$+\left[\left(\frac{28}{9}\gamma e-2\gamma^3 e-\frac{65}{9}\gamma e^3-\frac{14}{3}\gamma ee'^2\right)\frac{1}{m^2}-\frac{379}{18}\frac{\gamma e}{m}+\frac{16091}{432}\gamma e\right]\sin(\zeta+F-l)$$

(30) $$-\frac{5}{6}\frac{\gamma ee'}{m}\sin(\zeta+F-l-l')$$

(31) $+\frac{5}{6}\frac{\gamma ee'}{m}\sin(\zeta+F-l+l')$

(32) $+\left[-\frac{13}{4}\frac{\gamma e^2}{m^2}+\frac{809}{48}\frac{\gamma e^2}{m}\right]\sin(\zeta+F-2l)$

(33) $-\frac{79}{27}\frac{\gamma e^3}{m^3}\sin(\zeta+F-3l)$

(34) $-\left[\frac{2}{3}\frac{\gamma^3}{m^2}+\frac{1}{4}\frac{\gamma^3}{m}\right]\sin(\zeta+3F)$

(35) $-\frac{8}{3}\frac{\gamma^3 e}{m^2}\sin(\zeta+3F+l)$

(36) $-\frac{46}{9}\frac{\gamma^3 e}{m^2}\sin(\zeta+3F-l)$

(37) $$+\left[\left(\frac{38}{3}\gamma-7\gamma^3-\frac{20}{3}\gamma e^2-19\gamma e'^2+\frac{152}{9}\frac{\gamma}{m^2}\frac{f}{n}\right)\frac{1}{m^2}+\left(\frac{13}{4}\gamma-\frac{135}{8}\gamma^3-88\gamma e^2\right.\right.$$
$$\left.\left.-\frac{13}{9}\gamma e'^2\right)\frac{1}{m}+\frac{13585}{288}\gamma+\frac{5825}{576}\gamma m\right]\sin(\zeta-F)$$

(38) $+\left[\frac{9}{2}\frac{\gamma e'}{m}+\frac{3}{4}\gamma e'\right]\sin(\zeta-F-l')$

(39) $+\frac{27}{8}\frac{\gamma e'^2}{m}\sin(\zeta-F-2l')$

(40) $+\left[-\frac{9}{2}\frac{\gamma e'}{m}+\frac{57}{8}\gamma e'\right]\sin(\zeta-F+l')$

(41) $-\frac{27}{8}\frac{\gamma e'^2}{m}\sin(\zeta-F+2l')$

(42) $$+\left[\left(\frac{40}{3}\gamma e+\frac{45}{2}\gamma^3 e-\frac{80}{3}\gamma e^3-20\gamma ee'^2\right)\frac{1}{m^2}+\frac{53}{2}\frac{\gamma e}{m}+\frac{5929}{36}\gamma e\right]\sin(\zeta-F+l)$$

(43) $+\frac{91}{2}\frac{\gamma ee'}{m}\sin(\zeta-F+l-l')$

(44) $-\frac{91}{2}\frac{\gamma ee'}{m}\sin(\zeta-F+l+l')$

(45) $+\left[\frac{205}{12}\frac{\gamma e^2}{m^2}+\frac{255}{8}\frac{\gamma e^2}{m}\right]\sin(\zeta-F+2l)$

(46) $+\frac{45}{2}\frac{\gamma e^3}{m^2}\sin(\zeta-F+3l)$

(47) $$+\left[\left(\frac{40}{3}\gamma e+\frac{245}{6}\gamma^3 e-\frac{175}{6}\gamma e^3-20\gamma ee'^2\right)\frac{1}{m^2}+\frac{53}{2}\frac{\gamma e}{m}+\frac{2819}{18}\gamma e\right]\sin(\zeta-F-l)$$

(48) $-\frac{49}{2}\frac{\gamma ee'}{m}\sin(\zeta-F-l-l')$

(49) $$+\frac{49}{2}\frac{\gamma ee'}{m}\sin(\zeta - F - l + l')$$

(50) $$+\left[\frac{145}{9}\frac{\gamma e^2}{m^2} + \frac{1285}{36}\frac{\gamma e^2}{m}\right]\sin(\zeta - F - 2l)$$

(51) $$+\frac{185}{9}\frac{\gamma e^3}{m^2}\sin(\zeta - F - 3l)$$

(52) $$-\left[\frac{40}{3}\frac{\gamma^3}{m^2} + \frac{7}{2}\frac{\gamma^3}{m}\right]\sin(\zeta - 3F)$$

(53) $$-\frac{52}{3}\frac{\gamma^3 e}{m^2}\sin(\zeta - 3F + l)$$

(54) $$-40\frac{\gamma^3 e}{m^2}\sin(\zeta - 3F - l)$$

(55) $$+\left[\left(-\frac{3}{4}\gamma^3 + \frac{65}{8}\gamma e^2\right)\frac{1}{m} + \frac{11}{12}\gamma + \frac{1043}{288}\gamma m\right]\sin(\zeta + 2D + F)$$

(56) $$+\frac{77}{24}\gamma e'\sin(\zeta + 2D + F - l')$$

(57) $$-\frac{11}{24}\gamma e'\sin(\zeta + 2D + F + l')$$

(58) $$+\frac{13}{4}\gamma e\sin(\zeta + 2D + F + l)$$

(59) $$+\left[\frac{5}{2}\frac{\gamma e}{m} + \frac{2129}{144}\gamma e\right]\sin(\zeta + 2D + F - l)$$

(60) $$+\frac{35}{6}\frac{\gamma ee'}{m}\sin(\zeta + 2D + F - l - l')$$

(61) $$-\frac{5}{2}\frac{\gamma ee'}{m}\sin(\zeta + 2D + F - l + l')$$

(62) $$+\frac{245}{32}\frac{\gamma e^2}{m}\sin(\zeta + 2D + F - 2l)$$

(63) $$+\left[\left(\frac{1}{4}\gamma - \frac{65}{8}\gamma^3 + 62\gamma e^2 - \gamma e'^2\right)\frac{1}{m} + \frac{1775}{96}\gamma + \frac{161987}{2304}\gamma m\right]\sin(\zeta + 2D - F)$$

(64) $$+\left[\frac{7}{12}\frac{\gamma e'}{m} + \frac{6291}{96}\gamma e'\right]\sin(\zeta + 2D - F - l')$$

(65) $$+\frac{17}{16}\frac{\gamma e'^2}{m}\sin(\zeta + 2D - F - 2l')$$

(66) $$-\left[\frac{1}{4}\frac{\gamma e'}{m} + \frac{991}{96}\gamma e'\right]\sin(\zeta + 2D - F + l')$$

(67) $$-\frac{3}{16}\frac{\gamma e'^2}{m}\sin(\zeta + 2D - F + 2l')$$

(68) $$+\left[\frac{1}{2}\frac{\gamma e}{m}+\frac{2063}{48}\gamma e\right]\sin(\zeta+2D-F+l)$$

(69) $$+\frac{7}{6}\frac{\gamma ee'}{m}\sin(\zeta+2D-F+l-l')$$

(70) $$-\frac{1}{2}\frac{\gamma ee'}{m}\sin(\zeta+2D-F+l+l')$$

(71) $$+\frac{13}{16}\frac{\gamma e^2}{m}\sin(\zeta+2D-F+2l)$$

(72) $$+\left[\frac{49}{2}\frac{\gamma e}{m}+\frac{208}{3}\gamma e\right]\sin(\zeta+2D-F-l)$$

(73) $$+\frac{343}{6}\frac{\gamma ee'}{m}\sin(\zeta+2D-F-l-l')$$

(74) $$-\frac{49}{2}\frac{\gamma ee'}{m}\sin(\zeta+2D-F-l+l')$$

(75) $$-\frac{27}{16}\frac{\gamma e^2}{m}\sin(\zeta+2D-F-2l)$$

(76) $$-\frac{1}{4}\frac{\gamma^3}{m}\sin(\zeta+2D-3F)$$

(77) $$+\left[\left(\frac{3}{2}\gamma-3\gamma^3-\frac{75}{8}\gamma e^2-6\gamma e'^2\right)\frac{1}{m}-\frac{361}{96}\gamma+\frac{2357}{2304}\gamma m\right]\sin(\zeta-2D+F)$$

(78) $$-\left[\frac{3}{2}\frac{\gamma e'}{m}+\frac{263}{48}\gamma e'\right]\sin(\zeta-2D+F-l')$$

(79) $$-\frac{57}{16}\frac{\gamma e'^2}{m}\sin(\zeta-2D+F-2l')$$

(80) $$+\left[\frac{7}{2}\frac{\gamma e'}{m}-\frac{169}{24}\gamma e'\right]\sin(\zeta-2D+F+l')$$

(81) $$+\frac{51}{8}\frac{\gamma e'^2}{m}\sin(\zeta-2D+F+2l')$$

(82) $$+\left[\frac{\gamma e}{m}-\frac{217}{24}\gamma e\right]\sin(\zeta-2D+F+l)$$

(83) $$-\frac{\gamma ee'}{m}\sin(\zeta-2D+F+l-l')$$

(84) $$+\frac{7}{3}\frac{\gamma ee'}{m}\sin(\zeta-2D+F+l+l')$$

(85) $$-\frac{5}{4}\frac{\gamma e^2}{m}\sin(\zeta-2D+F+2l)$$

(86) $$+\left[-\frac{29}{6}\frac{\gamma e}{m}+\frac{1841}{144}\gamma e\right]\sin(\zeta-2D+F-l)$$

(87) $$+\frac{29}{6}\frac{\gamma ee'}{m}\sin(\zeta - 2D + F - l - l')$$

(88) $$-\frac{203}{18}\frac{\gamma ee'}{m}\sin(\zeta - 2D + F - l + l')$$

(89) $$-\frac{205}{32}\frac{\gamma e^2}{m}(\zeta - 2D + F - 2l)$$

(90) $$-2\frac{\gamma^3}{m}\sin(\zeta - 2D + 3F)$$

(91) $$+\left[\left(-\frac{1}{4}\gamma - \frac{87}{8}\gamma^3 + \frac{227}{4}\gamma e^2 + \gamma e'^2\right)\frac{1}{m} + \frac{65}{4}\gamma + \frac{6085}{96}\gamma m\right]\sin(\zeta - 2D - F)$$

(92) $$+\left[\frac{1}{4}\frac{\gamma e'}{m} - \frac{227}{32}\gamma e'\right]\sin(\zeta - 2D - F - l')$$

(93) $$+\frac{3}{8}\frac{\gamma e'^2}{m}\sin(\zeta - 2D - F - 2l')$$

(94) $$+\left[-\frac{7}{12}\frac{\gamma e'}{m} + \frac{5399}{96}\gamma e'\right]\sin(\zeta - 2D - F + l')$$

(95) $$-\frac{17}{16}\frac{\gamma e'^2}{m}\sin(\zeta - 2D - F + 2l')$$

(96) $$+\left[\frac{41}{2}\frac{\gamma e}{m} + \frac{2459}{48}\gamma e\right]\sin(\zeta - 2D - F + l)$$

(97) $$-\frac{41}{2}\frac{\gamma ee'}{m}\sin(\zeta - 2D - F + l - l')$$

(98) $$+\frac{287}{6}\frac{\gamma ee'}{m}\sin(\zeta - 2D - F + l + l')$$

(99) $$-\frac{19}{8}\frac{\gamma e^2}{m}\sin(\zeta - 2D - F + 2l)$$

(100) $$+\left[-\frac{1}{2}\frac{\gamma e}{m} + 39\gamma e\right]\sin(\zeta - 2D - F - l)$$

(101) $$+\frac{1}{2}\frac{\gamma ee'}{m}\sin(\zeta - 2D - F - l - l')$$

(102) $$-\frac{7}{6}\frac{\gamma ee'}{m}\sin(\zeta - 2D - F - l + l')$$

(103) $$-\frac{13}{16}\frac{\gamma e^2}{m}\sin(\zeta - 2D - F - 3l)$$

(104) $$+\frac{1}{4}\frac{\gamma^3}{m}\sin(\zeta - 2D - 3F)$$

(105) $$+\frac{11}{32}\gamma m\sin(\zeta + 4D - F)$$

(106) $+\frac{15}{16}\gamma e \sin(\zeta + 4D - F - l)$

(107) $+\left[-\frac{3}{32}\gamma + \frac{161}{128}\gamma m\right] \sin(\zeta - 4D + F)$

(108) $+\frac{3}{16}\gamma e' \sin(\zeta - 4D + F - l')$

(109) $-\frac{7}{16}\gamma e' \sin(\zeta - 4D + F + l')$

(110) $+\frac{33}{16}\gamma e \sin(\zeta - 4D + F + l)$

(111) $-\frac{3}{16}\gamma e \sin(\zeta - 4D + F - l)$

(112) $-\frac{11}{32}\gamma m \sin(\zeta - 4D - F)$

(113) $-\frac{15}{16}\gamma e \sin(\zeta - 4D - F + l)$

(114) $-\frac{5}{4}\frac{\gamma}{m}\frac{a}{a'} \sin(\zeta + D + F)$

(115) $+\frac{5}{3}\frac{\gamma e'}{m^3}\frac{a}{a'} \sin(\zeta + D + F + l')$

(116) $-\frac{25}{117}\frac{\gamma e e'}{m^3}\frac{a}{a'} \sin(\zeta + D + F - l + l')$

(117) $-\frac{75}{4}\frac{\gamma}{m}\frac{a}{a'} \sin(\zeta + D - F)$

(118) $+\left[\left(\frac{800}{1521}\gamma^3 e' + \frac{2000}{4563}\gamma e^2 e'\right)\frac{1}{m^4} + \frac{100}{117}\frac{\gamma e'}{m^3} + \frac{65945}{4563}\frac{\gamma e'}{m^3}\right]\frac{a}{a'} \sin(\zeta + D - F + l')$

(119) $+\frac{125}{117}\frac{\gamma e e'}{m^3}\frac{a}{a'} \sin(\zeta + D - F + l + l')$

(120) $-\frac{550}{117}\frac{\gamma e e'}{m^3}\frac{a}{a'} \sin(\zeta + D - F - l + l')$

(121) $+\frac{1000}{4563}\frac{\gamma e^2 e'}{m^4}\frac{a}{a'} \sin(\zeta + D - F - 2l + l')$

(122) $-\frac{25}{4}\frac{\gamma}{m}\frac{a}{a'} \sin(\zeta - D + F)$

(123) $+\frac{5}{3}\frac{\gamma e'}{m^3}\frac{a}{a'} \sin(\zeta - D + F - l')$

(124) $-5\frac{\gamma}{m}\frac{a}{a'} \sin(\zeta - D - F)$

(125) $+\frac{35}{3}\frac{\gamma e'}{m^3}\frac{a}{a'} \sin(\zeta - D - F - l')$

(126) $$-\frac{125}{78}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(\zeta - D - F + l')$$

(127) $$-\frac{25}{12}\frac{\gamma}{m}\frac{a}{a'}\sin(\zeta - 3D + F)\Big\}$$

(128) $$+\frac{\beta_3}{a^3}\Big\{\Big[\Big(-\frac{2}{3}\gamma^2 + \frac{14}{3}\gamma^4 - \frac{17}{3}\gamma^2 e^2 + \gamma^2 e'^2\Big)\frac{1}{m^2} - \frac{3}{4}\frac{\gamma^2}{m} + \frac{1}{12} - \frac{19}{18}\gamma^2 - \frac{3}{4}e^2$$

$$+\frac{65}{72}m^2\Big]\sin 2\zeta$$

(129) $$+\Big[-\frac{1}{2}\frac{\gamma^2 e'}{m} + \frac{1}{4}e'm\Big]\sin(2\zeta - l')$$

(130) $$+\Big[\frac{1}{2}\frac{\gamma^2 e'}{m} - \frac{1}{4}e'm\Big]\sin(2\zeta + l')$$

(131) $$-\Big[\frac{4}{3}\frac{\gamma^2 e}{m^2} + \frac{1}{2}\frac{\gamma^2 e}{m} - \frac{1}{6}e\Big]\sin(2\zeta + l)$$

(132) $$-\Big[\frac{13}{6}\frac{\gamma^2 e^2}{m^2} - \frac{13}{48}e^2\Big]\sin(2\zeta + 2l)$$

(133) $$+\Big[-\frac{16}{3}\frac{\gamma^2 e}{m^2} + 53\frac{\gamma^2 e}{m} - \frac{1}{6}e + \frac{625}{32}em\Big]\sin(2\zeta - l)$$

(134) $$+\Big[\frac{19}{2}\frac{\gamma^2 e^2}{m^2} + \frac{1}{16}e^2\Big]\sin(2\zeta - 2l)$$

(135) $$+\Big[\frac{2}{3}\frac{\gamma^4}{m^2} - \frac{1}{12}\gamma^2\Big]\sin(2\zeta + 2F)$$

(136) $$+\frac{5}{12}\frac{\gamma^2 e^2}{m^2}\sin(2\zeta + 2F - 2l)$$

(137) $$+\Big[\Big(-\frac{20}{3}\gamma^2 - \frac{22}{3}\gamma^4 + \frac{10}{3}\gamma^2 e^2 + 10\gamma^2 e'^2\Big)\frac{1}{m^2} - \frac{7}{4}\frac{\gamma^2}{m} - \frac{3869}{144}\gamma^2\Big]\sin(2\zeta - 2F)$$

(138) $$-\frac{9}{2}\frac{\gamma^2 e'}{m}\sin(2\zeta - 2F - l')$$

(139) $$+\frac{9}{2}\frac{\gamma^2 e'}{m}\sin(2\zeta - 2F + l')$$

(140) $$-\Big[\frac{20}{3}\frac{\gamma^2 e}{m^2} + \frac{53}{4}\frac{\gamma^2 e}{m}\Big]\sin(2\zeta - 2F + l)$$

(141) $$-\frac{35}{4}\frac{\gamma^2 e^2}{m^2}\sin(2\zeta - 2F + 2l)$$

(142) $$-\Big[\frac{20}{3}\frac{\gamma^2 e}{m^2} + \frac{53}{4}\frac{\gamma^2 e}{m}\Big]\sin(2\zeta - 2F - l)$$

(143) $$-\frac{15}{2}\frac{\gamma^2 e^2}{m^2}\sin(2\zeta - 2F - 2l)$$

(144) $+\frac{20}{3}\frac{\gamma^4}{m^3}\sin(2\zeta-4F)$

(145) $+\left[-\frac{11}{2}\gamma^2+\frac{19}{192}m^2\right]\sin(2\zeta+2D)$

(146) $+\left[-\frac{5}{2}\frac{\gamma^2 e}{m}+\frac{5}{16}em\right]\sin(2\zeta+2D-l)$

(147) $-\left[\frac{1}{4}\frac{\gamma^2}{m}+\frac{983}{96}\gamma^2\right]\sin(2\zeta+2D-2F)$

(148) $-\frac{7}{12}\frac{\gamma^2 e'}{m}\sin(2\zeta+2D-2F-l')$

(149) $+\frac{1}{4}\frac{\gamma^2 e'}{m}\sin(2\zeta+2D-2F+l')$

(150) $-\frac{1}{2}\frac{\gamma^2 e}{m}\sin(2\zeta+2D-2F+l)$

(151) $-\frac{29}{2}\frac{\gamma^2 e}{m}\sin(2\zeta+2D-2F-l)$

(152) $+\left[-\frac{3}{2}\frac{\gamma^2}{m}+\frac{221}{48}\gamma^2+\frac{1}{96}m^2\right]\sin(2\zeta-2D)$

(153) $+\frac{3}{2}\frac{\gamma^2 e'}{m}\sin(2\zeta-2D-l')$

(154) $-\frac{7}{2}\frac{\gamma^2 e'}{m}\sin(2\zeta-2D+l')$

(155) $-\left[\frac{\gamma^2 e}{m}+\frac{5}{32}em\right]\sin(2\zeta-2D+l)$

(156) $+\left[9\frac{\gamma^2 e}{m}+\frac{5}{32}em\right]\sin(2\zeta-2D-l)$

(157) $+\frac{3}{16}\gamma^2\sin(2\zeta-2D+2F)$

(158) $+\left[\frac{1}{4}\frac{\gamma^2}{m}-\frac{253}{32}\gamma^2\right]\sin(2\zeta-2D-2F)$

(159) $-\frac{1}{4}\frac{\gamma^2 e'}{m}\sin(2\zeta-2D-2F-l')$

(160) $+\frac{7}{12}\frac{\gamma^2 e'}{m}\sin(2\zeta-2D-2F+l')$

(161) $-\frac{21}{2}\frac{\gamma^2 e}{m}\sin(2\zeta-2D-2F+l)$

(162) $+\frac{1}{2}\frac{\gamma^2 e}{m}\sin(2\zeta-2D-2F-l)$

(163) $+\frac{3}{32}\gamma^3 \sin(2\zeta - 4D)$

(164) $+\frac{225}{64} em \sin(2\zeta - 4D + l)$

(165) $+\frac{45}{2}\frac{e'}{m}\frac{a}{a'} \sin(2\zeta - D - l')\Big\}.$

$U = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$

(1) $+\frac{\beta_1}{a^3}\Big\{\frac{8}{3}\gamma e \sin(F + l)$

(2) $+\left[\left(\frac{20}{3}\gamma^3 e - \frac{5}{3}\gamma e^3\right)\frac{1}{m^2} - 4\gamma e\right] \sin(F - l)$

(3) $+\left[\frac{5}{3}\frac{\gamma e^2}{m^2} - \frac{105}{8}\frac{\gamma e^2}{m}\right] \sin(F - 2l)$

(4) $+\frac{5}{3}\frac{\gamma e^3}{m^2} \sin(F - 3l)$

(5) $+\frac{20}{3}\frac{\gamma^3 e}{m^2} \sin(3F - l)$

(6) $+\frac{15}{4}\gamma e \sin(2D + F - l)$

(7) $-\frac{5}{8}\frac{\gamma e^2}{m} \sin(2D + F - 2l)$

(8) $-\left[\frac{3}{4}\gamma - \frac{1}{2}\gamma m\right] \sin(2D - F)$

(9) $-\frac{7}{6}\gamma e' \sin(2D - F - l')$

(10) $+\frac{3}{2}\gamma e' \sin(2D - F + l')$

(11) $+\frac{3}{4}\frac{\gamma e'^2}{m} \sin(2D - F + 2l')$

(12) $-\frac{3}{4}\gamma e \sin(2D - F + l)$

(13) $+\frac{9}{2}\gamma e \sin(2D - F - l)$

(14) $-\frac{10}{3}\frac{\gamma e'}{m^2}\frac{a}{a'} \sin(D + F + l')$

(15) $$-\frac{10}{3}\frac{\gamma e'}{m^2}\frac{a}{a'}\sin(\mathrm{D}-\mathrm{F}+l')\Big\}$$

(16) $$+\frac{\beta_2}{a^3}\Big\{\Big[\Big(-\frac{2}{3}+\frac{40}{3}\gamma^2+\frac{2}{3}e^2+e'^2-\frac{13}{2}\gamma^4+\frac{10}{3}\gamma^2e^2-\frac{267}{96}e^4-20\gamma^2e'^2-e^2e'^2$$
$$-\frac{1}{4}e'^4+\frac{5}{4}\frac{a^2}{a'^2}-\frac{8}{9}\frac{1}{m^2}\frac{f}{n}\Big)\frac{1}{m^2}+\Big(-\frac{1}{4}+\frac{9}{2}\gamma^2+6e^2+\frac{1}{9}e'^2-\frac{2}{3}\frac{1}{m^2}\frac{f}{n}\Big)\frac{1}{m}$$
$$-\frac{43}{18}+\frac{13223}{288}\gamma^2+\frac{30925}{1152}e^2-\frac{19}{4}e'^2-\frac{3449}{576}m-\frac{59245}{3456}m^2\Big]\sin\zeta$$

(17) $$+\Big[\Big(-\frac{1}{4}e'+11\gamma^2e'+\frac{5}{8}e^2e'+\frac{3}{32}e'^3\Big)\frac{1}{m}-\frac{3}{4}e'+\frac{1183}{96}e'm\Big]\sin(\zeta-l')$$

(18) $$-\Big[\frac{3}{16}\frac{e'^2}{m}+\frac{3}{32}e'^2\Big]\sin(\zeta-2l')$$

(19) $$-\frac{53}{288}\frac{e'^3}{m}\sin(\zeta-3l')$$

(20) $$+\Big[\Big(\frac{1}{4}e'-11\gamma^2e'-\frac{5}{8}e^2e'-\frac{3}{32}e'^3\Big)\frac{1}{m}+\frac{1}{4}e'-\frac{4757}{384}e'm\Big]\sin(\zeta+l')$$

(21) $$+\Big[\frac{3}{16}\frac{e'^2}{m}+\frac{85}{128}e'^2\Big]\sin(\zeta+2l')$$

(22) $$+\frac{53}{288}\frac{e'^3}{m}\sin(\zeta+3l')$$

(23) $$+\Big[\Big(-\frac{2}{3}e+\frac{80}{3}\gamma^2e+\frac{5}{6}e^3+ee'^2-\frac{8}{9}\frac{e}{m^2}\frac{f}{n}\Big)\frac{1}{m^2}+\Big(-\frac{1}{4}e+31\gamma^2e+\frac{97}{16}e^3$$
$$+\frac{1}{9}ee'^2\Big)\frac{1}{m}-\frac{14}{9}e-\frac{2909}{576}em\Big]\sin(\zeta+l)$$

(24) $$-\Big[2\frac{ee'}{m}+14ee'\Big]\sin(\zeta+l-l')$$

(25) $$-\frac{3}{2}\frac{ee'^2}{m}\sin(\zeta+l-2l')$$

(26) $$+\Big[2\frac{ee'}{m}+\frac{69}{8}ee'\Big]\sin(\zeta+l+l')$$

(27) $$+\frac{3}{2}\frac{ee'^2}{m}\sin(\zeta+l+2l')$$

(28) $$+\Big[\Big(-\frac{3}{4}e^2+\frac{545}{4}\gamma^2e^2+\frac{9}{8}e^4+\frac{9}{8}e^2e'^2\Big)\frac{1}{m^2}-\frac{9}{32}\frac{e^2}{m}-\frac{19}{16}e^2\Big]\sin(\zeta+2l)$$

(29) $$-\frac{135}{32}\frac{e^2e'}{m}\sin(\zeta+2l-l')$$

(30) $$+\frac{135}{32}\frac{e^2e'}{m}\sin(\zeta+2l+l')$$

(31) $$-\Big[\frac{8}{9}\frac{e^3}{m^3}+\frac{1}{3}\frac{e^3}{m}\Big]\sin(\zeta+3l)$$

(32) $$-\frac{625}{576}\frac{e^4}{m^2}\sin(\zeta+4l)$$

(33) $$+\left[\left(\frac{2}{3}e+\frac{40}{9}\gamma^2e-\frac{5}{9}e^3-ee'^2+\frac{8}{9}\frac{e}{m^2}\frac{f}{n}\right)\frac{1}{m^2}+\left(\frac{1}{4}e+\frac{13}{9}\gamma^2e-\frac{229}{72}e^3\right.\right.$$
$$\left.\left.-\frac{1}{9}ee'^2\right)\frac{1}{m}-\frac{167}{288}e+\frac{1345}{1152}em\right]\sin(\zeta-l)$$

(34) $$-\left[\frac{3}{2}\frac{ee'}{m}+\frac{41}{4}ee'\right]\sin(\zeta-l-l')$$

(35) $$-\frac{9}{8}\frac{ee'^2}{m}\sin(\zeta-l-2l')$$

(36) $$+\left[\frac{3}{2}\frac{ee'}{m}+\frac{83}{8}ee'\right]\sin(\zeta-l+l')$$

(37) $$+\frac{9}{8}\frac{ee'^2}{m}\sin(\zeta-l+2l')$$

(38) $$+\left[\left(\frac{23}{36}e^2+\frac{175}{12}\gamma^2e^2-\frac{13}{27}e^4-\frac{23}{24}e^2e'^2\right)\frac{1}{m^2}-\frac{731}{288}\frac{e^2}{m}+\frac{8045}{3456}e^2\right]\sin(\zeta-2l)$$

(39) $$-\frac{239}{96}\frac{e^2e'}{m}\sin(\zeta-2l-l')$$

(40) $$+\frac{239}{96}\frac{e^2e'}{m}\sin(\zeta-2l+l')$$

(41) $$+\left[\frac{11}{18}\frac{e^3}{m^2}-\frac{367}{144}\frac{e^3}{m}\right]\sin(\zeta-3l)$$

(42) $$+\frac{43}{64}\frac{e^4}{m^2}\sin(\zeta-4l)$$

(43) $$+\left[\left(\frac{1}{3}\gamma^2-\frac{19}{3}\gamma^4+\frac{43}{4}\gamma^2e^2-\frac{1}{2}\gamma^2e'^2\right)\frac{1}{m^2}+\frac{1}{8}\frac{\gamma^2}{m}-\frac{25}{18}\gamma^2\right]\sin(\zeta+2F)$$

(44) $$+\frac{3}{8}\frac{\gamma^2e'}{m}\sin(\zeta+2F-l')$$

(45) $$-\frac{3}{8}\frac{\gamma^2e'}{m}\sin(\zeta+2F+l')$$

(46) $$+\left[\frac{\gamma^2e}{m^2}+\frac{3}{8}\frac{\gamma^2e}{m}\right]\sin(\zeta+2F+l)$$

(47) $$+\frac{17}{8}\frac{\gamma^2e^2}{m^2}\sin(\zeta+2F+2l)$$

(48) $$+\left[\frac{46}{9}\frac{\gamma^2e}{m^2}-\frac{1867}{72}\frac{\gamma^2e}{m}\right]\sin(\zeta+2F-l)$$

(49) $$-\frac{43}{8}\frac{\gamma^2e^2}{m^2}\sin(\zeta+2F-2l)$$

(50) $-\frac{1}{4}\frac{\gamma^4}{m^3}\sin(\zeta+4F)$

(51) $+\left[\left(13\gamma^2-7\gamma^4-\frac{47}{4}\gamma^2e^2-\frac{39}{2}\gamma^2e'^2\right)\frac{1}{m^3}+\frac{27}{8}\frac{\gamma^2}{m}+\frac{4075}{96}\gamma^2\right]\sin(\zeta-2F)$

(52) $-\frac{15}{8}\frac{\gamma^2e'}{m}\sin(\zeta-2F-l')$

(53) $+\frac{15}{8}\frac{\gamma^2e'}{m}\sin(\zeta-2F+l')$

(54) $+\left[-\frac{4}{3}\frac{\gamma^2e}{m^3}+\frac{225}{8}\frac{\gamma^2e}{m}\right]\sin(\zeta-2F+l)$

(55) $+\frac{235}{24}\frac{\gamma^2e^2}{m^3}\sin(\zeta-2F+2l)$

(56) $+\left[\frac{79}{3}\frac{\gamma^2e}{m^3}+\frac{239}{8}\frac{\gamma^2e}{m}\right]\sin(\zeta-2F-l)$

(57) $+\frac{3193}{72}\frac{\gamma^2e^2}{m^3}\sin(\zeta-2F-2l)$

(58) $-\frac{79}{12}\frac{\gamma^4}{m^3}\sin(\zeta-4F)$

(59) $+\left[\left(\frac{1}{4}\gamma^2-\frac{45}{16}e^2\right)\frac{1}{m}-\frac{11}{24}+\frac{2743}{96}\gamma^2-\frac{1421}{128}e^2+\frac{11}{6}e'^2-\frac{505}{288}m-\frac{1313}{216}m^2\right]$
$\times\sin(\zeta+2D)$

(60) $+\left[\left(\frac{7}{12}\gamma^2e'-\frac{105}{16}e^2e'\right)\frac{1}{m}-\frac{77}{48}e'-\frac{4129}{384}e'm\right]\sin(\zeta+2D-l')$

(61) $-\frac{187}{48}e'^2\sin(\zeta+2D-2l')$

(62) $+\left[\left(-\frac{1}{4}\gamma^2e'+\frac{45}{16}e^2e'\right)\frac{1}{m}+\frac{11}{48}e'+\frac{2353}{1152}e'm\right]\sin(\zeta+2D+l')$

(63) $+\left[\left(\frac{3}{4}\gamma^2e-5e^3\right)\frac{1}{m}-\frac{7}{6}e-\frac{637}{144}em\right]\sin(\zeta+2D+l)$

(64) $-\frac{49}{12}ee'\sin(\zeta+2D+l-l')$

(65) $+\frac{7}{12}ee'\sin(\zeta+2D+l+l')$

(66) $-\frac{425}{192}e^2\sin(\zeta+2D+2l)$

(67) $+\left[\left(-\frac{5}{4}e+\frac{641}{12}\gamma^2e+\frac{5}{3}e^3+5ee'^2\right)\frac{1}{m}-\frac{527}{96}e-\frac{55499}{2304}em\right]\sin(\zeta+2D-l)$

(68) $-\left[\frac{35}{12}\frac{ee'}{m}+\frac{599}{32}ee'\right]\sin(\zeta+2D-l-l')$

(69) $$-\frac{85}{16}\frac{ee'^2}{m}\sin(\zeta+2D-l-2l')$$

(70) $$+\left[\frac{5}{4}\frac{ee'}{m}+\frac{241}{96}ee'\right]\sin(\zeta+2D-l+l')$$

(71) $$+\frac{15}{16}\frac{ee'^2}{m}\sin(\zeta+2D-l+2l)$$

(72) $$+\left[\frac{5}{48}\frac{e^2}{m}+\frac{1765}{1152}e^2\right]\sin(\zeta+2D-2l)$$

(73) $$+\frac{35}{144}\frac{e^2e'}{m}\sin(\zeta+2D-2l-l')$$

(74) $$-\frac{5}{48}\frac{e^2e'}{m}\sin(\zeta+2D-2l+l')$$

(75) $$-\frac{5}{6}\frac{e^3}{m}\sin(\zeta+2D-3l)$$

(76) $$+\frac{11}{16}\gamma^2\sin(\zeta+2D+2F)$$

(77) $$+\frac{15}{8}\frac{\gamma^2e}{m}\sin(\zeta+2D+2F-l)$$

(78) $$+\left[\frac{17}{4}\frac{\gamma^2}{m}+\frac{1253}{96}\gamma^2\right]\sin(\zeta+2D-2F)$$

(79) $$+\frac{119}{12}\frac{\gamma^2e'}{m}\sin(\zeta+2D-2F-l')$$

(80) $$-\frac{17}{4}\frac{\gamma^2e'}{m}\sin(\zeta+2D-2F+l')$$

(81) $$+\frac{85}{8}\frac{\gamma^2e}{m}\sin(\zeta+2D-2F+l)$$

(82) $$-\frac{3}{8}\frac{\gamma^2e}{m}\sin(\zeta+2D-2F-l)$$

(83) $$+\left[\left(\frac{1}{4}+\frac{21}{4}\gamma^2+\frac{9}{16}e^2-e'^2\right)\frac{1}{m}+\frac{17}{24}+\frac{469}{96}\gamma^2+\frac{69}{128}e^2-\frac{87}{16}e'^2+\frac{1289}{576}m\right.$$
$$\left.+\frac{5927}{864}m^2+\frac{1}{3}\frac{1}{m^3}\frac{f}{n}\right]\sin(\zeta-2D)$$

(84) $$+\left[\left(-\frac{1}{4}e'-\frac{21}{4}\gamma^2e'-\frac{9}{16}e^2e'+\frac{13}{32}e'^3\right)\frac{1}{m}-\frac{31}{24}e'-\frac{4015}{1152}e'm\right]\sin(\zeta-2D-l')$$

(85) $$+\left[-\frac{3}{8}\frac{e'^2}{m}-\frac{17}{32}e'^2\right]\sin(\zeta-2D-2l')$$

(86) $$-\frac{1}{96}\frac{e'^3}{m}\sin(\zeta-2D-3l')$$

(87) $$+\left[\left(\frac{7}{12}e' + \frac{49}{4}\gamma^2 e' + \frac{21}{16}e^2 e' - \frac{69}{32}e'^3\right)\frac{1}{m} + \frac{145}{48}e' + \frac{14183}{1152}e'm\right]\sin(\zeta - 2D + l')$$

(88) $$+\left[\frac{17}{16}\frac{e'^2}{m} + \frac{5917}{768}e'^2\right]\sin(\zeta - 2D + 2l')$$

(89) $$+\left[\left(e - \frac{17}{4}\gamma^2 e - \frac{1}{2}e^3 - 4ee'^2\right)\frac{1}{m} + \frac{149}{32}e + \frac{17401}{768}em\right]\sin(\zeta - 2D + l)$$

(90) $$-\left[\frac{ee'}{m} + \frac{7}{8}ee'\right]\sin(\zeta - 2D + l - l')$$

(91) $$-\frac{9}{16}\frac{ee'^2}{m}\sin(\zeta - 2D + l - 2l')$$

(92) $$+\left[\frac{7}{3}\frac{ee'}{m} + \frac{719}{48}ee'\right]\sin(\zeta - 2D + l + l')$$

(93) $$+\frac{17}{4}\frac{ee'^2}{m}\sin(\zeta - 2D + l + 2l')$$

(94) $$+\left[\frac{49}{32}\frac{e^2}{m} + \frac{3743}{384}e^2\right]\sin(\zeta - 2D + 2l)$$

(95) $$-\frac{49}{32}\frac{e^2e'}{m}\sin(\zeta - 2D + 2l - l')$$

(96) $$+\frac{343}{96}\frac{e^2e'}{m}\sin(\zeta - 2D + 2l + l')$$

(97) $$+\frac{67}{24}\frac{e^3}{m}\sin(\zeta - 2D + 3l)$$

(98) $$+\left[\left(\frac{1}{4}e + \frac{65}{12}\gamma^2 e + \frac{37}{24}e^3 - ee'^2\right)\frac{1}{m} + \frac{2}{3}e + \frac{265}{144}em\right]\sin(\zeta - 2D - l)$$

(99) $$-\left[\frac{1}{4}\frac{ee'}{m} + \frac{185}{96}ee'\right]\sin(\zeta - 2D - l - l')$$

(100) $$-\frac{3}{8}\frac{ee'^2}{m}\sin(\zeta - 2D - l - 2l')$$

(101) $$+\left[\frac{7}{12}\frac{ee'}{m} + \frac{113}{32}ee'\right]\sin(\zeta - 2D - l + l')$$

(102) $$+\frac{17}{16}\frac{ee'^2}{m}\sin(\zeta - 2D - l + 2l')$$

(103) $$+\left[\frac{9}{32}\frac{e^2}{m} + \frac{631}{576}e^2\right]\sin(\zeta - 2D - 2l)$$

(104) $$-\frac{9}{32}\frac{e^2e'}{m}\sin(\zeta - 2D - 2l - l')$$

(105) $$+\frac{21}{32}\frac{e^2e'}{m}\sin(\zeta - 2D - 2l + l')$$

(106) $$+\frac{1}{3}\frac{e^3}{m}\sin(\zeta - 2D - 3l)$$

(107) $$+\left[\frac{15}{8}\frac{\gamma^2}{m}-\frac{109}{32}\gamma^2\right]\sin(\zeta-2D+2F)$$

(108) $$-\frac{15}{8}\frac{\gamma^2 e'}{m}\sin(\zeta-2D+2F-l')$$

(109) $$+\frac{35}{8}\frac{\gamma^2 e'}{m}\sin(\zeta-2D+2F+l')$$

(110) $$+\frac{21}{4}\frac{\gamma^2 e}{m}\sin(\zeta-2D+2F+l)$$

(111) $$-\frac{139}{24}\frac{\gamma^2 e}{m}\sin(\zeta-2D+2F-l)$$

(112) $$+\left[-\frac{1}{8}\frac{\gamma^2}{m}+\frac{405}{16}\gamma^2\right]\sin(\zeta-2D-2F)$$

(113) $$+\frac{1}{8}\frac{\gamma^2 e'}{m}\sin(\zeta-2D-2F-l')$$

(114) $$-\frac{7}{24}\frac{\gamma^2 e'}{m}\sin(\zeta-2D-2F+l')$$

(115) $$-\frac{353}{8}\frac{\gamma^2 e}{m}\sin(\zeta-2D-2F+l)$$

(116) $$-\frac{3}{8}\frac{\gamma^2 e}{m}\sin(\zeta-2D-2F-l)$$

(117) $$-\frac{161}{384}m^2\sin(\zeta+4D)$$

(118) $$-\frac{35}{16}em\sin(\zeta+4D-l)$$

(119) $$-\frac{675}{256}e^2\sin(\zeta+4D-2l)$$

(120) $$+\frac{3}{64}\gamma^2\sin(\zeta+4D-2F)$$

(121) $$+\left[-\frac{3}{32}\gamma^2+\frac{135}{128}e^2+\frac{11}{64}m+\frac{15}{16}m^2\right]\sin(\zeta-4D)$$

(122) $$-\frac{33}{128}e'm\sin(\zeta-4D-l')$$

(123) $$+\frac{385}{384}e'm\sin(\zeta-4D+l')$$

(124) $$+\left[\frac{15}{32}e+\frac{401}{128}em\right]\sin(\zeta-4D+l)$$

(125) $$-\frac{15}{16}ee'\sin(\zeta-4D+l-l')$$

(126) $$+\frac{35}{16}ee'\sin(\zeta-4D+l+l')$$

(127) $$+\frac{195}{256}e^2\sin(\zeta-4D+2l)$$

(128) $$+\frac{7}{16}em\sin(\zeta-4D-l)$$

(129) $$+\frac{45}{16}\gamma^2\sin(\zeta-4D+2F)$$

(130) $$+\left[\frac{5}{8}\frac{1}{m}+\frac{709}{192}\right]\frac{a}{a'}\sin(\zeta+D)$$

(131) $$-\frac{5}{8}\frac{e'}{m}\frac{a}{a'}\sin(\zeta+D-l')$$

(132) $$+\left[\left(\frac{100}{117}\gamma^2e'+\frac{25}{117}e^2e'\right)\frac{1}{m^3}-\frac{5}{6}\frac{e'}{m^2}+\frac{55}{16}\frac{e'}{m}\right]\frac{a}{a'}\sin(\zeta+D+l')$$

(133) $$+\frac{45}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta+D+l)$$

(134) $$-\frac{15}{8}\frac{ee'}{m^2}\frac{a}{a'}\sin(\zeta+D+l+l')$$

(135) $$-\frac{15}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta+D-l)$$

(136) $$+\left[\left(\frac{400}{1521}\gamma^2ee'+\frac{250}{4563}e^3e'\right)\frac{1}{m^4}+\frac{25}{117}\frac{ee'}{m^3}-\frac{27815}{36504}\frac{ee'}{m^2}\right]\frac{a}{a'}\sin(\zeta+D-l+l')$$

(137) $$-\frac{25}{117}\frac{e^2e'}{m^3}\frac{a}{a'}\sin(\zeta+D-2l+l')$$

(138) $$+\frac{100}{117}\frac{\gamma^2e'}{m^3}\frac{a}{a'}\sin(\zeta+D-2F+l')$$

(139) $$+\frac{200}{1521}\frac{\gamma^2ee'}{m^4}\frac{a}{a'}\sin(\zeta+D-2F-l+l')$$

(140) $$-\left[\frac{5}{8}\frac{1}{m}+\frac{19}{8}\right]\frac{a}{a'}\sin(\zeta-D)$$

(141) $$+\left[\frac{5}{6}\frac{e'}{m^2}-\frac{55}{16}\frac{e'}{m}\right]\frac{a}{a'}\sin(\zeta-D-l')$$

(142) $$+\frac{5}{16}\frac{e'}{m}\frac{a}{a'}\sin(\zeta-D+l')$$

(143) $$-\frac{15}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta-D+l)$$

(144) $$+\frac{5}{8}\frac{ee'}{m^2}\frac{a}{a'}\sin(\zeta-D+l-l')$$

(145) $$-\frac{65}{32}\frac{e}{m}\frac{a}{a'}\sin(\zeta-D-l)$$

(146) $$+\frac{25}{24}\frac{ee'}{m^2}\frac{a}{a'}\sin(\zeta-D-l-l')$$

(147) $$-\frac{25}{312}\frac{ee'}{m^2}\frac{a}{a'}\sin(\zeta-D-l+l')$$

(148) $-\frac{5}{32}\frac{a}{a'}\sin(\zeta + 3D)$

(149) $-\frac{95}{192}\frac{a}{a'}\sin(\zeta - 3D)$

(150) $+\frac{5}{16}\frac{e'}{m}\frac{a}{a'}\sin(\zeta - 3D - l')$

(151) $-\frac{25}{48}\frac{e}{m}\frac{a}{a'}\sin(\zeta - 3D + l)\Big\}$

(152) $+\frac{\beta_3}{a^3}\Big\{-\left[\frac{1}{3}\frac{\gamma^3}{m^2} + \frac{1}{8}\frac{\gamma^3}{m} - \frac{1}{12}\gamma\right]\sin(2\zeta + F)$

(153) $-\left[\frac{\gamma^3 e}{m^2} + \frac{1}{4}\gamma e\right]\sin(2\zeta + F + l)$

(154) $+\left[-\frac{22}{3}\frac{\gamma^3 e}{m^2} - \frac{5}{12}\frac{\gamma e^3}{m^2} - \frac{1}{4}\gamma e\right]\sin(2\zeta + F - l)$

(155) $-\left[\frac{5}{12}\frac{\gamma e^2}{m^2} - \frac{85}{16}\frac{\gamma e^2}{m}\right]\sin(2\zeta + F - 2l)$

(156) $+\frac{5}{12}\frac{\gamma e^3}{m^3}\sin(2\zeta + F - 3l)$

(157) $+\left[\left(\frac{2}{3}\gamma - \frac{17}{3}\gamma^3 - \frac{2}{3}\gamma e^2 - \gamma e'^2\right)\frac{1}{m^2} + \left(\frac{1}{4}\gamma - \frac{23}{8}\gamma^3 - 6\gamma e^2 - \frac{1}{9}\gamma e'^2\right)\frac{1}{m}\right.$

$\left.+\frac{101}{36}\gamma + \frac{13481}{2304}\gamma m + \frac{8}{9}\frac{\gamma}{m^4}\frac{f}{n}\right]\sin(2\zeta - F)$

(158) $+\left[\frac{1}{4}\frac{\gamma e'}{m} + \frac{39}{16}\gamma e'\right]\sin(2\zeta - F - l')$

(159) $+\frac{3}{16}\frac{\gamma e'^2}{m}\sin(2\zeta - F - 2l')$

(160) $+\left[-\frac{1}{4}\frac{\gamma e'}{m} + \frac{23}{16}\gamma e'\right]\sin(2\zeta - F + l')$

(161) $-\frac{3}{16}\frac{\gamma e'^2}{m}\sin(2\zeta - F + 2l')$

(162) $+\left[\left(\frac{2}{3}\gamma e - \frac{37}{3}\gamma^3 e - \frac{5}{6}\gamma e^3 - \gamma e e'^2\right)\frac{1}{m^2} + \frac{1}{4}\frac{\gamma e}{m} + \frac{47}{36}\gamma e\right]\sin(2\zeta - F + l)$

(163) $+2\frac{\gamma e e'}{m}\sin(2\zeta - F + l - l')$

(164) $-2\frac{\gamma e e'}{m}\sin(2\zeta - F + l + l')$

(165) $+\left[\frac{3}{4}\frac{\gamma e^2}{m^2} + \frac{9}{32}\frac{\gamma e^2}{m}\right]\sin(2\zeta - F + 2l)$

(166) $+\frac{8}{9}\frac{\gamma e^3}{m^3}\sin(2\zeta - F + 3l)$

(167) $$+\left[-\left(\frac{2}{3}\gamma e+\frac{23}{3}\gamma^3 e-\frac{5}{6}\gamma e^3-\gamma e e'^2\right)\frac{1}{m^2}-\frac{1}{4}\frac{\gamma e}{m}+\frac{335}{288}\gamma e\right]\sin(2\zeta-F-l)$$

(168) $$+\frac{3}{2}\frac{\gamma e e'}{m}\sin(2\zeta-F-l-l')$$

(169) $$-\frac{3}{2}\frac{\gamma e e'}{m}\sin(2\zeta-F-l+l')$$

(170) $$+\left[-\frac{11}{12}\frac{\gamma e^2}{m^2}+\frac{209}{32}\frac{\gamma e^2}{m}\right]\sin(2\zeta-F-2l)$$

(171) $$-\frac{8}{9}\frac{\gamma e^3}{m^2}\sin(2\zeta-F-3l)$$

(172) $$-\left[\frac{19}{3}\frac{\gamma^3}{m^2}+\frac{17}{8}\frac{\gamma^3}{m}\right]\sin(2\zeta-3F)$$

(173) $$+\frac{4}{3}\frac{\gamma^3 e}{m^2}\sin(2\zeta-3F+l)$$

(174) $$-13\frac{\gamma^3 e}{m^2}\sin(2\zeta-3F-l)$$

(175) $$+\left[\left(-\frac{1}{4}\gamma^3+\frac{45}{16}\gamma e^2\right)\frac{1}{m}+\frac{11}{24}\gamma+\frac{1061}{576}\gamma m\right]\sin(2\zeta+2D-F)$$

(176) $$+\frac{77}{48}\gamma e'\sin(2\zeta+2D-F-l')$$

(177) $$-\frac{11}{48}\gamma e'\sin(2\zeta+2D-F+l')$$

(178) $$+\frac{7}{6}\gamma e\sin(2\zeta+2D-F+l)$$

(179) $$+\left[\frac{5}{4}\frac{\gamma e}{m}+\frac{527}{96}\gamma e\right]\sin(2\zeta+2D-F-l)$$

(180) $$+\frac{35}{12}\frac{\gamma e e'}{m}\sin(2\zeta+2D-F-l-l')$$

(181) $$-\frac{5}{4}\frac{\gamma e e'}{m}\sin(2\zeta+2D-F-l+l')$$

(182) $$-\frac{9}{4}\frac{\gamma^3}{m}\sin(2\zeta+2D-3F)$$

(183) $$+\left[-\frac{15}{8}\frac{\gamma^3}{m}-\frac{3}{16}\gamma+\frac{15}{64}\gamma m\right]\sin(2\zeta-2D+F)$$

(184) $$+\frac{3}{8}\gamma e'\sin(2\zeta-2D+F-l')$$

(185) $$-\frac{7}{24}\gamma e'\sin(2\zeta-2D+F+l')$$

(186) $-\frac{3}{16}\gamma e \sin(2\zeta - 2D + F + l)$

(187) $+\frac{3}{16}\gamma e \sin(2\zeta - 2D + F - l)$

(188) $+\frac{5}{32}\frac{\gamma e^2}{m} \sin(2\zeta - 2D + F - 2l)$

(189) $+\left[\left(-\frac{1}{4}\gamma - \frac{29}{8}\gamma^3 - \frac{9}{16}\gamma e^2 + \gamma e'^2\right)\frac{1}{m} - \frac{77}{96}\gamma - \frac{5291}{2304}\gamma m\right] \sin(2\zeta - 2D - F)$

(190) $+\left[\frac{1}{4}\frac{\gamma e'}{m} + \frac{31}{24}\gamma e'\right] \sin(2\zeta - 2D - F - l')$

(191) $+\frac{3}{16}\frac{\gamma e'^2}{m} \sin(2\zeta - 2D - F - 2l'$

(192) $-\left[\frac{7}{12}\frac{\gamma e'}{m} + \frac{19}{6}\gamma e'\right] \sin(2\zeta - 2D - F + l')$

(193) $-\frac{17}{16}\frac{\gamma e'^2}{m} \sin(2\zeta - 2D - F + 2l')$

(194) $-\left[\frac{\gamma e}{m} + \frac{73}{16}\gamma e\right] \sin(2\zeta - 2D - F + l)$

(195) $+\frac{\gamma e e'}{m} \sin(2\zeta - 2D - F + l - l')$

(196) $-\frac{7}{3}\frac{\gamma e e'}{m} \sin(2\zeta - 2D - F + l + l')$

(197) $-\frac{49}{32}\frac{\gamma e^2}{m} \sin(2\zeta - 2D - F + 2l)$

(198) $-\left[\frac{1}{4}\frac{\gamma e}{m} + \frac{37}{96}\gamma e\right] \sin(2\zeta - 2D - F - l)$

(199) $+\frac{1}{4}\frac{\gamma e e'}{m} \sin(2\zeta - 2D - F - l - l')$

(200) $-\frac{7}{12}\frac{\gamma e e'}{m} \sin(2\zeta - 2D - F - l + l')$

(201) $-\frac{9}{32}\frac{\gamma e^2}{m} \sin(2\zeta - 2D - F - 2l)$

(202) $+\frac{1}{8}\frac{\gamma^3}{m} \sin(2\zeta - 2D - 3F)$

(203) $+\frac{9}{256}\gamma m \sin(2\zeta - 4D + F)$

(204) $-\frac{11}{64}\gamma m \sin(2\zeta - 4D - F)$

(205) $\quad - \frac{15}{32} \gamma e \sin (2\zeta - 4D - F + l)$

(206) $\quad - \frac{5}{8} \frac{\gamma}{m} \frac{a}{a'} \sin (2\zeta + D - F)$

(207) $\quad + \frac{5}{6} \frac{\gamma e'}{m^2} \frac{a}{a'} \sin (2\zeta + D - F + l')$

(208) $\quad + \frac{5}{8} \frac{\gamma}{m} \frac{a}{a'} \sin (2\zeta - D - F)$

(209) $\quad - \frac{5}{6} \frac{\gamma e'}{m^2} \frac{a}{a'} \sin (2\zeta - D - F - l') \Big\}.$

(1) $\quad \frac{1}{r} = - \frac{1}{3} \frac{\beta_1}{a^3}$

(2) $\quad + \frac{\beta_2}{a^3} \Big\{ \frac{20}{9} \frac{\gamma e}{m^2} \cos (\zeta + F - l)$

(3) $\quad + \frac{20}{3} \frac{\gamma e}{m^2} \cos (\zeta - F + l)$

(4) $\quad - \frac{20}{3} \frac{\gamma e}{m^2} \cos (\zeta - F - l) \Big\}$

(5) $\quad - \frac{1}{6} \frac{\beta_3}{a^3} \cos 2\zeta.$

It remains to deduce the effect of the figure of the earth on the motions of the perigee and node. The terms left in R, after the 102 Operations have been performed, are

$$R = \beta_1 n^2 \Big\{ \frac{1}{3} - 2\gamma^2 + \frac{1}{2} e^2 + 2\gamma^4 - 3\gamma^2 e^2 + \frac{5}{8} e^4 + \left(- \frac{1}{2} + \frac{15}{2} \gamma^2 - \frac{9}{8} e^2 - \frac{3}{4} e'^2 \right) \frac{n'^2}{n^2}$$

$$+ \left(- \frac{51}{16} \gamma^2 + \frac{465}{64} e^2 \right) \frac{n'^3}{n^3} + \frac{79}{16} \frac{n'^4}{n^4} + \frac{421}{24} \frac{n'^5}{n^5} \Big\}$$

$$- \frac{9}{16} \beta_3 n^2 \gamma^2 e'^2 \frac{n'}{n} \cos (2\psi + 2h' + 2g').$$

On substituting this expression for R in the differential equations which give the motions of l, g, and h (p. 245), and making the transformation given (p. 246), and adding the terms arising from our Operation 102 in the values of $\frac{dl}{dt}$, $\frac{dg}{dt}$, and $\frac{dh}{dt}$, given by DELAUNAY (Vol. II, pp. 237, 238), we get the following equations:

$$\frac{d(g+h)}{dt} = \frac{\beta_1}{a^2} n \left[1 - 8\gamma^2 + 2e^2 + \frac{827}{96} m^2 + \frac{3405}{64} m^3 \right],$$

$$\frac{dh}{dt} = - n \Big\{ \frac{\beta_1}{a^2} \left[1 - 2\gamma^2 + 2e^2 + \frac{35}{96} m^2 + \frac{3}{64} m^3 \right] + \frac{9}{32} \frac{\beta_3}{a^2} e'^2 m \cos (2\psi + 2h' + 2g') \Big\}.$$

These expressions are correct to terms of the eighth order inclusive.

CHAPTER V.

DISCUSSION OF PENDULUM EXPERIMENTS WITH THE OBJECT OF DETERMINING THE VALUE OF THE FACTOR, TO WHICH ARE PROPORTIONAL THE PERTURBATIONS OF THE MOON PRODUCED BY THE FIGURE OF THE EARTH.

In this chapter we propose to derive the value of the constant factor

$$\frac{3}{2}\frac{1}{MD^2}\left(C - \frac{A+B}{2}\right)$$

from the measures of the intensity of gravity made at stations on the earth's surface. It is essential to the success of the treatment that the measures be supposed to belong to a level surface; what one is immaterial, provided we know its dimensions from geodetic measures. As many of the measures have been made at, or a very short distance above, sea level, it will be advantageous to select sea level as the level surface to be employed. Then all the measures which have not been made at sea level ought to be reduced to what they would have been had the pendulum been swung at a point where the vertical, through the station, meets the level of the sea, brought in by a tunnel.

D represents a length which is nearly the average of the equatorial radii of sea level, and it will be taken as the equivalent of the distance to which belongs the constant of the moon's equatorial horizontal parallax.

The portion of the earth's mass, which lies outside of this level surface, is somewhere about 100000th of the whole; and its influence in determining the proper form of the development of the potential function of the earth's mass may be neglected. We consequently assume that this function can be expanded in an infinite series proceeding according to negative integral powers of the distance from the center of gravity.

Let ρ denote the earth's density at the point x', y', z', and T the duration of a revolution of the earth on its axis, then V the potential of gravity, centrifugal force being included, is given by the expression

$$V = \iiint \frac{\rho dx' dy' dz'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}} + \frac{2\pi^2}{T^2}(x^2 + y^2).$$

The triple integral must be extended to all points of the earth's mass; after which the variables x', y', and z' disappear, and V becomes a function of x, y, and z, which, equated to a constant, gives the general equation to level surfaces. Let c be the special value of this constant which belongs to the level surface of the sea. V can then be partially

differentiated with respect to x, y, and z; and if g denote the force of gravity at a point of the sea level, whose geographical latitude and longitude are, respectively, φ' and ω', we shall have, simultaneously, the four equations

$$\begin{aligned} c &= \mathrm{V}, \\ g \cos\varphi' \cos\omega' &= -\frac{d\mathrm{V}}{dx}, \\ g \cos\varphi' \sin\omega' &= -\frac{d\mathrm{V}}{dy}, \\ g \sin\varphi' &= -\frac{d\mathrm{V}}{dz}. \end{aligned}$$

If the variables x, y, and z are eliminated from these four equations a single equation will be left, giving a relation between the variables g, φ', and ω', which, being solved with reference to g, affords the value of g in terms of φ' and ω'.

To facilitate this elimination, we introduce polar co-ordinates in place of x, y, and z, such that

$$\begin{aligned} x &= r\cos\varphi\cos\omega, \\ y &= r\cos\varphi\sin\omega, \\ z &= r\sin\varphi; \end{aligned}$$

thus φ and ω are the geocentric latitude and longitude of the point x, y, z. Then our four equations take the form

$$\begin{aligned} c = \mathrm{V} &= \iiint \frac{\rho\, dx'dy'dz'}{[r^2 - 2r(x'\cos\varphi\cos\omega + y'\cos\varphi\sin\omega + z'\sin\varphi) + r'^2]^{\frac{1}{2}}} + \frac{2\pi^2}{\mathrm{T}^2} r^2 \cos^2\varphi, \\ \frac{d\mathrm{V}}{dr} &= -g\,[\cos\varphi'\cos\varphi\cos(\omega' - \omega) + \sin\varphi'\sin\varphi], \\ \frac{1}{r}\frac{d\mathrm{V}}{d\varphi} &= -g\,[-\cos\varphi'\sin\varphi\cos(\omega' - \omega) + \sin\varphi'\cos\varphi], \\ \frac{1}{r\cos\varphi}\frac{d\mathrm{V}}{d\omega} &= -g\cos\varphi'\sin(\omega' - \omega). \end{aligned}$$

From these the variables r, φ, and ω must be eliminated. Practically the variable ω may be eliminated in the following manner. The difference $\omega' - \omega$ between the geographical and geocentric longitude probably nowhere exceeds a minute of arc; consequently we can put $\cos(\omega' - \omega) = 1$; and in the development of the first part of V, it is known that the terms, which involve ω, have very small coefficients; hence, in these, it will be allowable to substitute ω' for ω. In this way ω disappears, and our equations are reduced to the three:

$$\begin{aligned} c = \mathrm{V} &= \iiint \frac{\rho\, dx'dy'dz'}{[r^2 - 2r(x'\cos\varphi\cos\omega' + y'\cos\varphi\sin\omega' + z\sin\varphi) + r'^2]^{\frac{1}{2}}} + \frac{2\pi^2}{\mathrm{T}^2} r^2 \cos^2\varphi, \\ \frac{d\mathrm{V}}{dr} &= -g\cos(\varphi' - \varphi), \\ \frac{1}{r}\frac{d\mathrm{V}}{d\varphi} &= -g\sin(\varphi' - \varphi). \end{aligned}$$

We shall now suppose that the first part of V is expanded in a series of spherical functions; and, employing LAPLACE's notation, let it be sufficient to stop with Y_4. Our three equations may then be written

$$\frac{M}{r} + \frac{Y_2}{r^3} + \frac{Y_3}{r^4} + \frac{Y_4}{r^5} + \frac{2\pi^2}{T^2} r^2 \cos^2 \varphi \qquad = c,$$

$$\frac{M}{r^2} + 3\frac{Y_2}{r^4} + 4\frac{Y_3}{r^5} + 5\frac{Y_4}{r^6} - 4\frac{\pi^2}{T^2} r \cos^2 \varphi \qquad = g \cos(\varphi - \varphi'),$$

$$\frac{1}{r^4}\frac{dY_2}{d\varphi} + \frac{1}{r^5}\frac{dY_3}{d\varphi} + \frac{1}{r^6}\frac{dY_4}{d\varphi} - 4\frac{\pi^2}{T^2} r \sin \varphi \cos \varphi = g \sin(\varphi - \varphi').$$

To facilitate the elimination of r and φ from these equations, we square both members of the first and divide by M. Here the squares and product of Y_3 and Y_4, as well as their product by the last term of the first member, may be neglected. This gives

$$\frac{M}{r^2} + 2\frac{Y_2}{r^4} + 2\frac{Y_3}{r^5} + 2\frac{Y_4}{r^6} + \frac{4\pi^2}{T^2} r \cos^2 \varphi = \frac{c^2}{M} - \frac{1}{M}\frac{Y_2^2}{r^6} - \frac{4\pi^2}{MT^2}\frac{Y_2 \cos^2 \varphi}{r} - \frac{4\pi^4}{MT^4} r^4 \cos^4 \varphi.$$

By subtracting this from the second equation we get

$$\frac{c^2}{M} + \frac{Y_2}{r^4} + 2\frac{Y_3}{r^5} + 3\frac{Y_4}{r^6} = g\cos(\varphi - \varphi') + \frac{8\pi^2}{T^2} r \cos^2 \varphi + \frac{1}{M}\frac{Y_2^2}{r^6} + \frac{4\pi^2}{MT^2}\frac{Y_2 \cos^2 \varphi}{r} + \frac{4\pi^4}{MT^4} r^4 \cos^4 \varphi.*$$

It will be noticed that, in this equation, wherever the variables r and φ occur, they are multiplied by quantities which are, at least, of the order of smallness of the compression. Hence it will suffice to eliminate them by formulæ which are only approximately exact. For this purpose we assume that the meridian is an ellipse; and taking the compression at $\frac{1}{294.98}$, the formulæ, by which r and φ may be eliminated, are

$$r = D(1 - 0.0034096 \sin^2 \varphi + 0.0000195 \sin^4 \varphi),$$

$$\varphi = \varphi' - 700''.44 \sin 2\varphi' + 3''.79 \sin 4\varphi'.$$

In making the computations we assume D as the linear unit; according to LISTING its value in meters is a number whose common logarithm is 6.8046421.† We adopt T as the unit of time; thus the logarithm of the number, by which the length of the second's pendulum in meters ought to be multiplied to produce the value of g corresponding to these units, is 4.0603104. Sufficiently approximate values of M and Y_2, for computing the value of the right member of our equation, are given by the equations

$$\log M = \quad 4.0571257,$$

$$Y_2 \quad = -18.8196\left(\sin^2 \varphi - \frac{1}{3}\right).$$

*The last term of the second member of this equation, of the order of the square of the compression, was inadvertently omitted in the numerical discussion which follows. The found value of H_1 ought, in consequence, to be corrected by the addition of the quantity $\delta H_1 = -0.0387$.

†Astr. Nachr. Band 93, s. 317.

Substituting for Y_2, Y_3, and Y_4 their known values in terms of φ and ω', and employing N to denote the right member of the equation, which is a known quantity, we have

$$\left.\begin{array}{ll}
\quad H_0 & + H_{10} r^{-5} \cos^3 \varphi \sin 3\omega' \\
+ H_1 r^{-4}\left(\sin^2 \varphi - \frac{1}{3}\right) & + H_{11} r^{-6}\left(\sin^4 \varphi - \frac{6}{7}\sin^2 \varphi + \frac{3}{35}\right) \\
+ H_2 r^{-4} \cos^2 \varphi \cos 2\omega' & + H_{12} r^{-6}\left(\sin^3 \varphi - \frac{3}{7}\sin \varphi\right)\cos \varphi \cos \omega' \\
+ H_3 r^{-4} \cos^2 \varphi \sin 2\omega' & + H_{13} r^{-6}\left(\sin^3 \varphi - \frac{3}{7}\sin \varphi\right)\cos \varphi \sin \omega' \\
+ H_4 r^{-5}\left(\sin^3 \varphi - \frac{3}{5}\sin \varphi\right) & + H_{14} r^{-6}\left(\sin^2 \varphi - \frac{1}{7}\right)\cos^2 \varphi \cos 2\omega' \\
+ H_5 r^{-5}\left(\sin^2 \varphi - \frac{1}{5}\right)\cos \varphi \cos \omega' & + H_{15} r^{-6}\left(\sin^2 \varphi - \frac{1}{7}\right)\cos^2 \varphi \sin 2\omega' \\
+ H_6 r^{-5}\left(\sin^2 \varphi - \frac{1}{5}\right)\cos \varphi \sin \omega' & + H_{16} r^{-6} \sin \varphi \cos^3 \varphi \cos 3\omega' \\
+ H_7 r^{-5} \sin \varphi \cos^2 \varphi \cos 2\omega' & + H_{17} r^{-6} \sin \varphi \cos^3 \varphi \sin 3\omega' \\
+ H_8 r^{-5} \sin \varphi \cos^2 \varphi \sin 2\omega' & + H_{18} r^{-6} \cos^4 \varphi \cos 4\omega' \\
+ H_9 r^{-5} \cos^3 \varphi \cos 3\omega' & + H_{19} r^{-6} \cos^4 \varphi \sin 4\omega'
\end{array}\right\} = N.$$

Here $H_0 \ldots H_{19}$ denote a series of constants, not necessarily having any dependence on each other, and which must be determined from observation. For our present purpose we require only the value of H_1, the equivalent of which is

$$\frac{3}{2}\left(C - \frac{A + B}{2}\right) = -H_1.$$

In order to have only small quantities to deal with, we assume as approximate values of H_0 and H_1,

$$H_0 = \frac{c^2}{M} = 11458.574,$$

$$H_1 = -18.8196;$$

and then subtract from N the correspondent value of the two first terms of the first member. H_0 and H_1 can then be replaced by δH_0 and δH_1, the corrections of the assumed values of H_0 and H_1, and N by δN.

A collection of the results of pendulum experiments has been made by Dr. A. FISCHER,* and we avail ourselves of it for the present discussion. The data are given in the following table. The longitudes of the stations are counted from Paris, and the length of the second's pendulum is in meters.

*Astr. Nachr. Band 88, s. 81.

Results of Pendulum Experiments.

Station.	φ′.			ω′.		Length of Second's Pendulum.	Obs.—Cal.
	°	′	″	°	′	m.	
1. Spitzbergen	+79	49	58	− 9	40	0.9960373	+0.0000562
2. Melville	74	47	12	+113	8	0.9958398	+0.0000427
3. Greenland	74	32	19	+ 21	20	0.9957484	−0.0000598
4. Port Bowen	73	13	39	+ 91	15	0.9957428	+0.0000045
5. Hammerfest	70	40	5	− 21	25	0.9955276	−0.0000337
6. Kandalaks	67	7	43	− 30	6	0.9953298	−0.0000014
7. Drontheim	63	25	54	− 8	3	0.9950095	−0.0000979
8. Unst	60	45	28	+ 3	11	0.9949348	+0.0000225
9. Petersburg	59	56	31	− 27	58	0.9948640	+0.0000324
10. Stockholm	59	21	0	− 15	40	0.9947837	−0.0000057
11. Portsoy	57	40	59	+ 5	5	0.9946886	+0.0000272
12. Sitka	57	2	58	+137	40	0.9945948	−0.0000568
13. Leith Fort	55	58	41	+ 5	35	0.9945348	+0.0000191
14. Königsberg	54	42	50	− 8	10	0.9944098	+0.0000032
15. Güldenstein	54	13	6	− 8	30	0.9943860	+0.0000218
16. Altona	53	32	45	− 7	36	0.9943270	+0.0000217
17. Clifton	53	27	43	+ 3	33	0.9942921	−0.0000015
18. Petropaulowsk	53	0	59	−156	23	0.9943250	−0.0000969
19. Berlin	52	30	17	− 11	4	0.9942318	+0.0000151
20. Arbury Hill	52	12	55	+ 3	33	0.9942047	+0.0000229
21. Leyden	52	9	20	− 2	9	0.9942072	+0.0000280
22. London	51	31	8	+ 2	26	0.9941200	+0.0000010
23. Greenwich	51	28	40	+ 2	20	0.9941177	+0.0000023
24. Dunkirk	51	2	10	0	0	0.9940805	+0.0000038
25. Gotha	50	56	38	− 8	23	0.9939856	−0.0000912
26. Seeberg	50	56	6	− 8	28	0.9940655	−0.0000107
27. Inselberg	50	51	11	− 8	8	0.9940746	+0.0000064
28. Bonn	50	43	45	− 4	46	0.9940689	+0.0000155
29. Shanklin Farm	50	37	24	+ 3	32	0.9940370	+0.0000001
30. Mannheim	49	29	11	− 6	8	0.9939027	−0.0000404
31. Paris	48	50	14	0	0	0.9938510	−0.0000257
32. Clermont	45	46	48	− 0	46	0.9935848	−0.0000131
33. Milan	45	28	1	− 6	51	0.9935476	−0.0000352
34. Padua	45	24	3	− 9	32	0.9936073	+0.0000235
35. Fiume	45	19	0	− 12	48	0.9935841	−0.0000008
36. Bordeaux	44	50	26	+ 2	54	0.9934550	−0.0000501
37. Figeac	44	36	45	+ 0	17	0.9934603	−0.0000286
38. Toulon	43	7	20	− 3	36	0.9933644	+0.0000012
39. Barcelona	41	23	15	+ 0	12	0.9932321	+0.0000356
40. New York	40	42	43	+ 76	20	0.9931555	−0.0000065
41. Formentera	38	39	56	+ 0	55	0.9929755	+0.0000239
42. Lipari	38	28	37	− 12	33	0.9930792	+0.0000872
43. Bonin Islands	27	4	12	−140	0	0.9923284	+0.0001487
44. San Blas	21	32	24	+107	36	0.9915627	−0.0000807
45. Mowi	20	52	7	+159	2	0.9917632	+0.0000201
46. Jamaica	17	56	7	+ 79	10	0.9914677	+0.0000124
47. Guam	13	26	18	−142	26	0.9913800	−0.0000658

Results of Pendulum Experiments—Continued.

Station.	φ'.			ω'.		Length of Second's Pendulum.	Obs.—Cal.
	°	′	″	°	′	m.	
48. Madras	+13	4	9	− 77	57	0.9911857	−0.0000217
49. Trinidad	10	38	55	+ 63	51	0.9910677	−0.0000369
50. Porto Bello	9	32	30	+ 81	57	0.9911775	+0.0000780
51. Sierra Leone	8	29	28	+ 15	39	0.9910743	−0.0000403
52. Ualan	5	21	16	−160	41	0.9912605	−0.0000242
53. Galapagos	0	32	19	+ 92	50	0.9910057	+0.0000279
54. St. Thomas	0	24	41	− 4	24	0.9911043	−0.0000641
55. Pulo Gaumah	+ 0	1	49	−139	3	0.9910537	−0.0000021
56. Rawak	− 0	1	34	−128	35	0.9909345	−0.0000065
57. Para	1	27	0	+ 50	49	0.9909155	−0.0000098
58. Maranham	2	31	35	+ 46	36	0.9908720	−0.0000721
59. Fernando de Noronha	3	49	59	+ 34	43	0.9911582	+0.0001525
60. Ascension	7	55	23	+ 16	44	0.9911830	+0.0000058
61. Bahia	12	59	21	+ 41	51	0.9911857	−0.0000374
62. St. Helena	15	56	7	+ 8	3	0.9915515	+0.0000519
63. Isle de France	20	9	23	− 55	8	0.9917650	+0.0000342
64. Rio de Janeiro	22	55	22	+ 45	30	0.9917030	−0.0000215
65. Valparaiso	33	2	30	+ 74	2	0.9924741	−0.0000291
66. Paramatta	33	48	43	−148	40	0.9925441	−0.0000090
67. Port Jackson	33	51	34	−148	0	0.9925907	+0.0000310
68. Cape of Good Hope	33	54	37	− 16	8	0.9925410	+0.0000163
69. Montevideo	34	54	26	+ 58	33	0.9926105	+0.0000034
70. Falkland Islands	51	31	44	+ 60	28	0.9941164	−0.0000031
71. Staten Island	54	46	23	+ 66	21	0.9944702	+0.0000342
72. Cape Horn	55	51	20	+ 69	53	0.9945340	−0.0000144
73. South Shetland	−62	56	11	+ 62	54	0.9951450	+0.0000475

The equations of condition, which result from these observations for the determination of $H_0 \ldots H_{19}$, are given below; I have preferred to give the logarithms of the coefficients, but the absolute terms are numbers.

Equations of Condition.

	δH_1.	H_4.	H_{11}.	H_2.	H_3.	H_5.	H_6.	H_7.	H_8.	H_9.
1.	9.8085	9.5665	9.2953	8.4799	8.0250_n	9.1362	8.3675_n	8.4743	8.0194_n	7.6980
2.	9.7814	9.5101	9.1949	8.6887_n	8.7079_n	8.8863_n	9.2557	8.6744_n	8.6936_n	8.2434
3.	9.7798	9.5065	9.1886	8.7292	8.6938	9.2668	8.8585	8.7143	8.6789	7.9344
4.	9.7706	9.4869	9.1520	8.9310_n	7.5711_n	7.6633_n	9.3244	8.9132_n	7.5533_n	7.2114
5.	9.7501	9.4421	9.0652	8.9156	8.8827_n	9.3362	8.9298_n	8.8914	8.8585_n	8.2120
6.	9.7158	9.3633	8.8962	8.8856	9.1276_n	9.3465	9.1097_n	8.8509	9.0929_n	6.5016_n
7.	9.6716	9.2529	8.5914	9.2932	8.7535_n	9.4310	8.5816_n	9.2453	8.7056_n	8.9249
8.	9.6334	9.1467	8.0520	9.3840	8.4316	9.4433	8.1886	9.3252	8.3728	9.0729
9.	9.6206	9.1080	7.4980	9.1566	9.3265_n	9.3911	9.1162_n	9.0942	9.2641_n	8.1377

Equations of Condition—Continued.

	δH_1.	H_4.	H_{11}.	H_2.	H_3.	H_5.	H_6.	H_7.	H_8.	H_9.
10.	9.6109	9.0775	7.4044$_n$	9.3550	9.1395$_n$	9.4289	8.8767$_n$	9.2899	9.0744$_n$	8.9680
11.	9.5818	8.9785	8.2616$_n$	9.4576	8.7113	9.4425	8.3917	9.3847	8.6344	9.1801
12.	9.5700	8.9341	8.3809$_n$	8.4477	9.4775$_n$	9.3117$_n$	9.2712	8.3717	9.4015$_n$	8.9975
13.	9.5490	8.8469	8.5220$_n$	9.4954	8.7907	9.4377	8.4279	9.4140	8.7093	9.2358
14.	9.5224	8.7165	8.6384$_n$	9.5133	8.9802$_n$	9.4299	8.5868$_n$	9.4251	8.8920$_n$	9.2548
15.	9.5114	8.6534	8.6742$_n$	9.5222	9.0075$_n$	9.4270	8.6015$_n$	9.4313	8.9166$_n$	9.2669
16.	9.4959	8.5507	8.7183$_n$	9.5400	8.9741$_n$	9.4240	8.5492$_n$	9.4453	8.8794$_n$	9.2970
17.	9.4939	8.5366	8.7242$_n$	9.5538	8.6491	9.4264	8.2190	9.4587	8.5540	9.3272
18.	9.4832	8.4497	8.7483$_n$	9.3980	9.4319	9.3863$_n$	9.0270$_n$	9.3003	9.3342	8.8642$_n$
19.	9.4705	8.3223	8.7750$_n$	9.5430	9.1523$_n$	9.4125	8.7039$_n$	9.4423	9.0516$_n$	9.2860
20.	9.4632	8.2312	8.7887$_n$	9.5785	8.6738	9.4173	8.2099	9.4762	8.5715	9.3643
21.	9.4616	8.2100	8.7923$_n$	9.5818	8.4579$_n$	9.4177	7.9922$_n$	9.4791	8.3552$_n$	9.3708
22.	9.4449	7.8796	8.8203$_n$	9.5936	8.5238	9.4122	8.0405	9.4870	8.4172	9.3883
23.	9.4438	7.8465	8.8216$_n$	9.5945	8.5063	9.4119	8.0220	9.4877	8.3995	9.3898
24.	9.4317	7.0348	8.8391$_n$	9.6042	. . . .	9.4082	. . . .	9.4946	. . . .	9.4053
25.	9.4291	6.3347$_n$	8.8416$_n$	9.5871	9.0660$_n$	9.4027	8.5711$_n$	9.4769	8.9558$_n$	9.3647
26.	9.4288	6.5106$_n$	8.8423$_n$	9.5869	9.0704$_n$	9.4023	8.5751$_n$	9.4766	8.9601$_n$	9.3641
27.	9.4265	7.1464$_n$	8.8454$_n$	9.5899	9.0549$_n$	9.4020	8.5571$_n$	9.4791	8.9441$_n$	9.3698
28.	9.4230	7.4933$_n$	8.8498$_n$	9.6038	8.8289$_n$	9.4039	8.3250$_n$	9.4924	8.7175$_n$	9.4003
29.	9.4200	7.6529$_n$	8.8535$_n$	9.6085	8.7018	9.4035	8.1941	9.4963	8.5896	9.4093
30.	9.3860	8.2867$_n$	8.8899$_n$	9.6221	8.9594$_n$	9.3897	8.4209$_n$	9.5026	8.8399$_n$	9.4245
31.	9.3652	8.4425$_n$	8.9065$_n$	9.6434	. . . .	9.3846	. . . .	9.5197	. . . .	9.4643
32.	9.2507	8.8109$_n$	8.9650$_n$	9.6928	8.1205$_n$	9.3405	7.4670$_n$	9.5475	7.9752$_n$	9.5384
33.	9.2371	8.8341$_n$	8.9688$_n$	9.6852	9.0722$_n$	9.3320	8.4116$_n$	9.5375	8.9245$_n$	9.5173
34.	9.2341	8.8389$_n$	8.9692$_n$	9.6743	9.2129$_n$	9.3279	8.5530$_n$	9.5261	9.0647$_n$	9.4909
35.	9.2303	8.8445$_n$	8.9706$_n$	9.6552	9.3357$_n$	9.3216	8.6780$_n$	9.5063	9.1868$_n$	9.4435
36.	9.2084	8.8765$_n$	8.9761$_n$	9.7050	8.7118	9.3233	8.0280	9.5525	8.5593	9.5551
37.	9.1974	8.8906$_n$	8.9788$_n$	9.7106	7.7058	9.3197	7.0139	9.5563	7.5515	9.5652
38.	9.1185	8.9703$_n$	8.9895$_n$	9.7286	8.8301$_n$	9.2885	8.0873$_n$	9.5625	8.6640$_n$	9.5895
39.	9.0045	9.0425$_n$	8.9950$_n$	9.7555	7.5994	9.2485	6.7914	9.5748	7.4187	9.6327
40.	8.9509	9.0658$_n$	8.9945$_n$	9.7129$_n$	9.4263	8.6039	9.2180	9.5263$_n$	9.2397	9.4627$_n$
41.	8.7329	9.1248$_n$	8.9866$_n$	9.7895	8.2947	9.1686	7.3728	9.5840	8.0892	9.6835
42.	8.7063	9.1294$_n$	8.9857$_n$	9.7488	9.4194$_n$	9.1517	8.4992$_n$	9.5415	9.2121$_n$	9.5858
43.	9.1099$_n$	9.2540$_n$	8.6821$_n$	9.1413	9.8950	7.5357$_n$	7.4595$_n$	8.7974	9.5511	9.5511
44.	9.3020$_n$	9.2323$_n$	8.0346$_n$	9.8510$_n$	9.6995$_n$	8.2750	8.7737$_n$	9.4135$_n$	9.2620$_n$	9.8091
45.	9.3187$_n$	9.2264$_n$	7.7864$_n$	9.8140	9.7674$_n$	8.8148	8.3981$_n$	9.3634	9.3168$_n$	9.5721$_n$
46.	9.3801$_n$	9.1909$_n$	8.1531	9.9260$_n$	9.5251	8.2799$_n$	8.9981$_n$	9.4120$_n$	9.0111	9.6668$_n$
47.	9.4475$_n$	9.1016$_n$	8.6319	9.3857	9.9617	9.0539	8.9399	8.7493	9.3253	9.5512
48.	9.4519$_n$	9.0919$_n$	8 6526	9.9381$_n$	9.5888$_n$	8.4836$_n$	9.1542	9.2899$_n$	8.9405$_n$	9.7375$_n$
49.	9.4768$_n$	9.0170$_n$	8.7630	9.7717$_n$	9.8836	8.8578$_n$	9.1668$_n$	9.0355$_n$	9.1474	9.9690$_n$
50.	9.4862$_n$	8.9748$_n$	8.8009	9.9708$_n$	9.4312	8.3782$_n$	9.2276$_n$	9.1875$_n$	8.6479	9.5941$_n$
51.	9.4940$_n$	8.9288$_n$	8.8314	9.9224	9.7063	9.2307$_n$	8.6780$_n$	9.0888	8.8727	9.8203
52.	9.5116$_n$	8.7389$_n$	8.8944	9.8890	9.7917	9.2549	8.7996	8.8561	8.7588	9.7193$_n$
53.	9.5228$_n$	7.7482$_n$	8.9325	9.9979$_n$	8.9945$_n$	7.9948	9.3003$_n$	7.9681$_n$	6.9647$_n$	9.1696
54.	9.5228$_n$	7.6313$_n$	8.9330	9.9948	9.1847$_n$	9.2996$_n$	8.1858	7.8480	7.0379$_n$	9.9883
55.	9.5229$_n$	6.4971$_n$	8.9330	9.1489	9.9957	9.1791	9.1175	5.8679	6.7147	9.7344
56.	9.5229$_n$	6.4322	8.9330	9.3466$_n$	9.9890	9.0959	9.1940	6.0007	6.6431$_n$	9.9546
57.	9.5221$_n$	8.1779	8.9304	9.3043$_n$	9.9907	9.1002$_n$	9.1890$_n$	7.7046	8.3910$_n$	9.9473$_n$

Equations of Condition—Continued.

	δH_1.	H_4.	H_{11}.	H_2.	H_3.	H_5.	H_6.	H_7.	H_8.	H_9.
58.	9.5204$_n$	8.4184	8.9238	8.7460$_n$	9.9985	9.1334$_n$	9.1577$_n$	7.3877	8.6402$_n$	9.8818$_n$
59.	9.5171$_n$	8.5971	8.9133	9.5438	9.9695	9.2051$_n$	9.0458$_n$	8.3660$_n$	8.7917$_n$	9.3853$_n$
60.	9.4978$_n$	8.9009	8.8451	9.9132	9.7334	9.2355$_n$	8.7136$_n$	9.0497$_n$	8.8699$_n$	9.7941
61.	9.4528$_n$	9.0897	8.6584	9.0184	9.9755	9.0378$_n$	8.9899$_n$	8.3674$_n$	9.3245$_n$	9.7316$_n$
62.	9.4136$_n$	9.1571	8.4385	9.9494	9.4098	9.0783$_n$	8.2289$_n$	9.3855$_n$	8.8459$_n$	9.9104
63.	9.3352$_n$	9.2192	7.0424$_n$	9.4860$_n$	9.9188$_n$	8.6483$_n$	8.8051	9.0209	9.9537	9.9053$_n$
64.	9.2642$_n$	9.2423	8.3109$_n$	8.1722$_n$	9.9302	8.5106$_n$	8.5181$_n$	7.7604	9.5184$_n$	9.7558$_n$
65.	8.5913$_n$	9.2217	8.9057$_n$	9.7790$_n$	9.5737	8.3411	8.8846	9.5139	9.3086$_n$	9.6454$_n$
66.	8.4261$_n$	9.2128	8.9223$_n$	9.5047	9.7913	8.8827$_n$	8.6672$_n$	9.2486$_n$	9.5352$_n$	8.6072
67.	8.4134$_n$	9.2123	8.9233$_n$	9.4840	9.7959	8.8823$_n$	8.6781$_n$	9.2284$_n$	9.5403$_n$	8.7819
68.	8.3995$_n$	9.2116	8.9244$_n$	9.7688	9.5691$_n$	8.9396	8.4008$_n$	9.5138$_n$	9.3141	9.5842
69.	7.9484$_n$	9.1980	8.9434$_n$	9.4900$_n$	9.7810	8.7299	8.9434	9.2461	9.5371$_n$	9.7455$_n$
70.	9.4451	7.8860$_n$	8.8190$_n$	9.3060$_n$	9.5284	9.1055	9.3523	9.1994	9.4218$_n$	9.3915$_n$
71.	9.5236	8.7241$_n$	8.6333$_n$	9.3613$_n$	9.3962	9.0379	9.3965	9.2735	9.3084$_n$	9.2694$_n$
72.	9.5465	8.8355$_n$	8.5336$_n$	9.3892$_n$	9.3166	8.9758	9.4119	9.3071	9.2345$_n$	9.1977$_n$
73.	9.6650	9.2353$_n$	8.5281	9.0924$_n$	9.2344	9.0960	9.3870	9.0426	9.1846$_n$	8.9818$_n$

	H_{10}.	H_{12}.	H_{13}.	H_{14}.	H_{15}.	H_{16}.	H_{17}.	H_{18}.	H_{19}.	$-\delta N$.
1.	7.4418$_n$	8.9774	8.2087$_n$	8.3996	7.9447$_n$	7.6925	7.4363$_n$	6.8998	6.8030$_n$	−0.241
2.	7.8184$_n$	8.7088$_n$	9.0782	8.5876$_n$	8.6068$_n$	8.2290	7.8040$_n$	6.3406$_n$	7.6948	−0.230
3.	8.2463	9.0883	8.6800	8.6269	8.5915	7.9196	8.2315	6.6330	7.7212	+0.689
4.	8.3949$_n$	7.4784$_n$	9.1395	8.8217$_n$	7.4618$_n$	7.1936	8.3771$_n$	7.8583	6.8003	+0.021
5.	8.5287$_n$	9.1370	8.7306$_n$	8.7910	8.7581$_n$	8.1878	8.5045$_n$	6.9763	8.0968$_n$	+0.917
6.	8.7826$_n$	9.1226	8.8858$_n$	8.7359	8.9779$_n$	6.4668$_n$	8.7478$_n$	8.0801$_n$	8.3117$_n$	+0.704
7.	8.5765$_n$	9.1739	8.3245$_n$	9.1118	8.5721$_n$	8.8770	8.5286$_n$	8.5462	8.3454$_n$	+1.447
8.	8.2988	9.1562	7.9015	9.1758	8.2234	9.0141	8.2400	8.7604	8.1144	−0.006
9.	9.1088$_n$	9.0934	8.8185$_n$	8.9397	9.1096$_n$	8.0753	9.0464$_n$	8.3854$_n$	8.7820$_n$	+0.074
10.	8.9983$_n$	9.1233	8.5711$_n$	9.1314	8.9159$_n$	8.9030	8.9333$_n$	8.5068	8.7934$_n$	+0.456
11.	8.6157	9.1123	8.0615	9.2144	8.4681	9.1072	8.5428	8.8990	8.4678	−0.005
12.	9.1203	8.9714$_n$	8.9309	8.1967	9.2265$_n$	8.9215	9.0443	8.9490$_n$	8.2240$_n$	+0.471
13.	8.7143	9.0786	8.0688	9.2309	8.5262	9.1543	8.6328	8.9715	8.5852	+0.129
14.	8.9135$_n$	9.0465	8.2034$_n$	9.2314	8.6983$_n$	9.1666	8.8253$_n$	8.9857	8.7927$_n$	+0.328
15.	8.9454$_n$	9.0331	8.2076$_n$	9.2336	8.7189$_n$	9.1760	8.8545$_n$	8.9999	8.8289$_n$	+0.111
16.	8.9206$_n$	9.0152	8.1404$_n$	9.2418	8.6759$_n$	9.2023	8.8259$_n$	9.0448	8.8132$_n$	+0.119
17.	8.6015	9.0156	7.8082	9.2544	8.3497	9.2321	8.5064	9.0989	8.5021	+0.436
18.	9.3235$_n$	8.9648$_n$	8.6055$_n$	9.0919	9.1258	8.7665$_n$	9.2258$_n$	8.0217$_n$	9.1290	−0.389
19.	9.1018$_n$	8.9783	8.2697$_n$	9.2293	8.8386$_n$	9.1854	9.0012$_n$	9.0056	8.9942$_n$	+0.166
20.	8.6386	8.9755	7.7681	9.2604	8.3557	9.2616	8.5359	9.1483	8.5515	+0.184
21.	8.4241$_n$	8.9740	7.5485$_n$	9.2626	8.1387$_n$	9.2680	8.3213$_n$	9.1593	8.3389$_n$	+0.095
22.	8.4958	8.9510	7.5793	9.2644	8.1946	9.2817	8.3892	9.1822	8.4165	+0.448
23.	8.4789	8.9495	7.5596	9.2647	8.1765	9.2830	8.3721	9.1843	8.4001	+0.434
24.	. . . .	8.9327	. . . .	9.2672	. . . .	9.2958	. . . .	9.2066	. . . .	+0.409
25.	9.0362$_n$	8.9242	8.0926$_n$	9.2487	8.7276$_n$	9.2546	8.9261$_n$	9.1309	8.9523$_n$	+1.406
26.	9.0406$_n$	8.9237	8.0965$_n$	9.2483	8.7318$_n$	9.2539	8.9304$_n$	9.1295	8.9564$_n$	+0.479
27.	9.0264$_n$	8.9207	8.0758$_n$	9.2499	8.7149$_n$	9.2591	8.9157$_n$	9.1392	8.9440$_n$	+0.290
28.	8.8066$_n$	8.9187	7.8398$_n$	9.2619	8.4870$_n$	9.2888	8.6951$_n$	9.1934	8.7320$_n$	+0.228

Equations of Condition—Continued.

	H_{10}.	H_{12}.	H_{13}.	H_{14}.	H_{15}.	H_{16}.	H_{17}.	H_{18}.	H_{19}.	$-\delta N$.
29.	8.6815	8.9148	7.7054	9.2649	8.3582	9.2972	8.5694	9.2085	8.6095	+0.487
30.	8.9465_n	8.8613	7.8925_n	9.2591	8.5964_n	9.3051	8.8271_n	9.2215	8.8[illegible]09$_n$	+0.858
31.	. . . .	8.8302	. . . .	9.2689	. . . .	9.3405	. . . .	9.2851	. . . .	+0.780
32.	8.1423_n	8.6152	6.7417_n	9.2595	7.6872_n	9.3930	7.9969_n	9.3839	8.1128_n	+0.652
33.	9.0912_n	8.5824	7.6620_n	9.2453	8.6323_n	9.3696	8.9435_n	9.3424	9.0571_n	+0.751
34.	9.2275_n	8.5729	7.7980_n	9.2330	8.7716_n	9.3427	9.0793_n	9.2919	9.1869_n	−0.004
35.	9.3426_n	8.5593	7.9157_n	9.2121	8.8926_n	9.2947	9.1938_n	9.1957	9.2904_n	+0.175
36.	8.7398	8.5178	7.2225	9.2518	8.2586	9.4027	8.5874	9.4040	8.7164	+1.158
37.	7.7365	8.4903	6.1845	9.2524	7.2476	9.4109	7.5822	9.4196	7.7159	+0.859
38.	8.8700_n	8.2491	7.0479_n	9.2365	8.3380_n	9.4234	8.7039_n	9.4487	8.8583_n	+0.402
39.	7.6527	7.4160	4.9589	9.2206	7.0645	9.4519	7.4719	9.5098	7.6548	+0.115
40.	9.5236_n	6.8812_n	7.4953_n	9.1601_n	8.8735	9.2760_n	9.3369_n	9.2896	9.4390_n	+0.293
41.	8.3650	8.3086_n	6.5128_n	9.1784	7.6836	9.4779	8.1594	9.5771	8.3841	+0.256
42.	9.4731_n	8.3298_n	7.6773	9.1320	8.8026_n	9.3785	9.2658_n	9.3890	9.4682_n	−1.127
43.	9.7896_n	8.8417	8.7655	7.9345	8.6882	9.2072	9.4457_n	9.7757_n	9.3368	−3.305
44.	9.6893_n	8.4833	8.9820_n	7.8349	7.6834	9.3716	9.2518_n	9.4026	9.8511	+1.155
45.	9.8631	8.9732	8.5565_n	8.0563_n	8.0097	9.1215_n	9.4125	8.9134	9.8822_n	−1.624
46.	9.8626_n	8.2641_n	8.9823_n	8.6181	8.2172_n	9.1527_n	9.3485_n	9.7772	9.7519_n	−0.154
47.	9.9297_n	8.8255	8.7115	8.3379_n	8.9139_n	8.9148	9.2933_n	9.8916_n	9.6484	−1.597
48.	9.8738	8.2381_n	8.9087	8.9039	8.5545	9.0891_n	9.2254	9.7792	9.8278	+0.463
49.	9.2794_n	8.4972_n	8.8062_n	8.8098	8.9217_n	9.2329_n	8.5433_n	9.3720_n	9.9562_n	+0.798
50.	9.9425_n	7.9607_n	8.8101_n	9.0344	8.4948_n	8.8107_n	9.1591_n	9.9037	9.7029_n	−0.862
51.	9.8498	8.7551_n	8.2024_n	9.0065_n	8.7904_n	8.9867	9.0162	9.6443	9.9296	−0.018
52.	9.9227_n	8.5632	8.1079	9.0171_n	8.9198_n	8.6863_n	8.8897_n	9.3358	9.9818	−2.944
53.	9.9951_n	6.2961	7.6016_n	9.1525	8.1492	7.1398	7.9653_n	9.9913	9.2933	−0.534
54.	9.3586_n	7.4838_n	6.3700	9.1495_n	8.3394	7.8415	7.2118_n	9.9792	9.4805_n	−1.669
55.	9.9243_n	6.2291	6.1675	8.3038_n	9.1506_n	6.4534	6.6433_n	9.9824_n	9.4456	−1.091
56.	9.6379_n	6.0810_n	6.1791_n	8.5015	9.1439_n	6.6087_n	6.2920	9.9549_n	9.6366_n	+0.279
57.	9.6647	7.8322	7.9210	8.4573	9.1437_n	8.3476	8.0650_n	9.9626_n	9.5961_n	+0.535
58.	9.8087	8.1083	8.1326	7.8950	9.1475_n	8.5235	8.4504_n	9.9956_n	9.0455_n	+1.113
59.	9.9837	8.3635	8.2042	8.6851_n	9.1108_n	8.2075	8.8059_n	9.8731_n	9.8143	−2.022
60.	9.8733	8.7264	8.2045	9.0071_n	8.8273_n	8.9307_n	9.0099_n	9.5769	9.9476	−1.434
61.	9.8775	8.7886	8.7407	7.9869_n	8.9440_n	9.0805	9.2264_n	9.9454_n	9.2947	+0.426
62.	9.5620	8 9646	8.1152	8.7849_n	8.2453_n	9.3464_n	8.9980_n	9.8609	9.6601	−2.284
63.	9.3211_n	8.7589	8.9157_n	7.8933	8.3261	9.4401	8.8559	9.7734_n	9.7054	−2.133
64.	9.7330	8.8447	8.8522	6.0246_n	7.7826	9.3440	9.3212_n	9.8599_n	8.4030_n	+0.557
65.	9.6013_n	8.2282	8.7717	8.9605_n	8.7552	9.3803	9.3362	9.3435	9.6527_n	+0.434
66.	9.7625_n	8.6836_n	8.4681_n	8.7202	9.0068	8.3510_n	9.5063	9.4467_n	9.5961	+0.373
67.	9.7603_n	8.6778_n	8.4736_n	8.7015	9.0134	8.5262_n	9.5046	9.4728_n	9.5800	−0.116
68.	9.6359_n	8.7297	8.1909_n	8.9884	8.7887_n	9.3291_n	9.3808	9.3160	9.6381_n	+0.504
69.	8.6267	8.4081	8.6216	8.7502_n	9.0412	9.5016	8.3828_n	9.4293_n	9.5713_n	+0.678
70.	7.7795_n	8.6445_n	8.8913_n	8.9770_n	9.1994	9.2850	7.6731	8.8616_n	9.1335_n	+0.502
71.	8.8076_n	8.6557_n	9.0143_n	9.0804_n	9.1153	9.1816	8.7198	7.9621_n	9.0565_n	−0.307
72.	8.9530_n	8.6145_n	9.0506_n	9.1229_n	9.0503	9.1156	8.8709	8.2299	9.0048_n	+0.019
73.	8.1665_n	8.8337_n	9.1247_n	8.9063_n	9.0483	8.9320	8.1167	8.1475_n	8.6255_n	−0.526

Attributing equal weights to these equations of condition, the normal equations, derived from them by the method of least squares are as follows:

Normal Equations.

$73.000\,\delta H_0$	$+8.022\,\delta H_1$	$+1.850\,H_4$	$-0.886\,H_{11}$	$+12.665\,H_2$	$+10.146\,H_3$	$+7.908\,H_5$
+ 8.022	+7.2322	+1.3430	−0.3117	+ 2.7542	− 3.0720	+2.8059
+ 1.850	+1.3430	+1.2292	+0.2816	+ 0.3351	+ 0.1790	+0.1358
− 0.886	−0.3117	+0.2816	+0.4329	− 0.8077	+ 0.4708	−0.4245
+12.665	+2.7542	+0.3351	−0.8077	+15.8556	+ 0.2961	+2.5739
+10.146	−3.0720	+0.1790	+0.4708	+ 0.2961	+13.3376	−0.7115
+ 7.908	+2.8059	+0.1358	−0.4245	+ 2.5739	− 0.7115	+2.5592
+ 0.212	+0.5819	+0.0018	−0.0150	− 0.2765	− 0.0615	+0.0746
+ 9.597	+2.9145	−0.0967	−0.6771	+ 4.4313	− 0.4369	+2.4667
− 3.274	−0.7005	−0.3194	+0.1429	− 0.4370	− 0.4739	−0.3258
+ 5.814	+1.8527	−0.3164	−0.5027	+ 8.2193	− 1.9308	+1.8676
− 4.922	−0.3284	+0.3704	+0.2913	+ 2.4849	− 0.3736	−1.3554
+ 3.115	+0.7927	+0.2583	−0.0414	+ 1.1072	− 0.0625	+0.7818
− 0.460	−0.0944	+0.1245	+0.0435	+ 0.1548	+ 0.1908	−0.1360
+ 5.167	+1.3844	+0.0277	−0.3448	+ 1.7351	− 0.5767	+1.3310
− 1.130	+0.2028	−0.0637	−0.1142	− 0.5767	− 0.3907	−0.1283
+ 7.133	+1.6009	−0.0249	−0.4879	+ 2.1056	+ 0.1531	+1.4808
− 1.615	−0.0631	+0.2064	−0.0170	− 0.0179	− 0.7154	−0.2596
+ 2.505	+1.6433	−0.6846	−0.4614	+ 1.7185	− 6.9137	+1.0194
+ 0.598	−1.2652	+0.2150	+0.3268	+ 2.4860	+ 1.1438	−0.6468

$+0.212\,H_6$	$+9.597\,H_7$	$-3.274\,H_8$	$+5.814\,H_9$	$-4.922\,H_{10}$	$+3.115\,H_{12}$	$-0.460\,H_{13}$
+0.5819	+2.9145	−0.7005	+ 1.8527	− 0.3284	+0.7927	−0.0944
+0.0018	−0.0967	−0.3194	− 0.3164	+ 0.3704	+0.2583	+0.1245
−0.0150	−0.6771	+0.1429	− 0.5027	+ 0.2913	−0.0414	+0.0435
−0.2765	+4.4313	−0.4370	+ 8.2193	+ 2.4849	+1.1072	+0.1548
−0.0615	−0.4369	−0.4739	− 1.9308	− 0.3736	−0.0625	+0.1908
+0.0746	+2.4667	−0.3258	+ 1.8676	− 1.3554	+0.7818	−0.1360
+0.7367	+0.0470	−0.0934	+ 0.1688	− 0.1130	−0.1352	+0.0493
+0.0470	+4.0039	−0.5357	+ 2.1055	− 0.0179	+0.6777	−0.0668
−0.0934	−0.5357	+1.5163	+ 0.1532	− 0.7157	−0.0670	+0.0472
+0.1688	+2.1055	+0.1532	+12.1432	− 2.2143	+0.5135	+0.0529
−0.1130	−0.0179	−0.7157	− 2.2143	+11.5669	+0.0111	+0.3550
−0.1352	+0.6777	−0.0670	+ 0.5135	+ 0.0111	+0.4045	−0.0083
+0.0493	−0.0668	+0.0472	+ 0.0529	+ 0.3550	−0.0083	+0.17327
−0.3181	+1.5613	−0.0925	+ 1.2352	− 0.5835	+0.4347	−0.0309
+0.2042	−0.0925	−0.0026	− 0.2756	− 0.1549	−0.1194	−0.0456
+0.1247	+2.4114	−0.5510	+ 1.3293	− 0.3078	+0.3978	−0.0235
+0.1988	−0.2288	−0.2091	− 0.3078	+ 0.5153	−0.1594	+0.0077
−0.2388	+0.9878	+0.1014	+ 4.4284	− 1.8728	+0.1259	−0.2191
+0.1211	−0.7161	+0.1734	+ 2.2228	+ 1.2293	+0.0096	+0.1829

Normal Equations—Continued.

$+5.167\ H_{14}$	$-1.130\ H_{15}$	$+7.133\ H_{16}$	$-1.615\ H_{17}$	$+\ 2.505\ H_{18}$	$+0.598\ H_{19}$	$+0.1210 = 0,$
$+1.3844$	$+0.2028$	$+1.6009$	-0.0631	$+\ 1.6433$	-1.2652	$+8.6530 = 0,$
$+0.0277$	-0.0637	-0.0249	$+0.2064$	$-\ 0.6846$	$+0.2150$	$+1.3522 = 0,$
-0.3448	-0.1142	-0.4879	-0.0170	$-\ 0.4614$	$+0.3268$	$-1.3453 = 0,$
$+1.7351$	-0.5767	$+2.1056$	-0.0179	$+\ 1.7185$	$+2.4860$	$-4.9310 = 0,$
-0.5767	-0.3907	$+0.1531$	-0.7154	$-\ 6.9137$	$+1.1438$	$-4.8670 = 0,$
$+1.3310$	-0.1283	$+1.4808$	-0.2596	$+\ 1.0194$	-0.6468	$+3.6864 = 0,$
-0.3181	$+0.2042$	$+0.1247$	$+0.1988$	$-\ 0.2388$	$+0.1211$	$-0.5356 = 0,$
$+1.5613$	-0.0925	$+2.4114$	-0.2288	$+\ 0.9878$	-0.7161	$+3.1106 = 0,$
-0.0925	-0.0026	-0.5510	-0.2091	$+\ 0.1014$	$+0.1734$	$-2.9294 = 0,$
$+1.2352$	-0.2756	$+1.3293$	-0.3078	$+\ 4.4284$	$+2.2228$	$-1.2000 = 0,$
-0.5835	-0.1549	-0.3078	$+0.5153$	$-\ 1.8728$	$+1.2293$	$+4.3510 = 0,$
$+0.4347$	-0.1194	$+0.3978$	-0.1594	$+\ 0.1259$	$+0.0096$	$+0.4696 = 0,$
-0.0309	-0.0456	-0.0235	$+0.0077$	$-\ 0.2191$	$+0.1829$	$-0.1673 = 0,$
$+1.1804$	-0.1622	$+0.9390$	-0.1593	$+\ 0.8672$	-0.4319	$+3.1603 = 0,$
-0.1622	$+0.3815$	-0.0116	$+0.1876$	$+\ 0.5214$	-0.0131	$+0.0113 = 0,$
$+0.9390$	-0.0116	$+1.9771$	-0.2717	$+\ 0.0797$	$+0.1063$	$+2.6599 = 0,$
-0.1593	$+0.1876$	-0.2717	$+1.0134$	$+\ 0.1068$	-0.0067	$+1.2388 = 0,$
$+0.8672$	$+0.5214$	$+0.0797$	$+0.1068$	$+12.4342$	-0.2535	$+0.8415 = 0,$
-0.4319	-0.0131	$+0.1063$	-0.0067	$-\ 0.2535$	$+8.3059$	$-7.9150 = 0.$

The equations derived from these in the process of solution are

$$\left\{\begin{array}{l} +2.5232\delta H_0 +1.6047\delta H_1 -\ 0.6780H_4 -0.4514H_{11}+1.7944H_2 -6.8788H_3 +0.9997H_5 -0.2351H_6 \\ +0.9659\ H_7 +0.1067\ H_8 +\ 4.4962H_9 -1.8353H_{10}+0.1262H_{12}-0.2135H_{13}+0.8540H_{14}+0.5210H_{15} \\ +0.0829\ H_{16}+0.1066\ H_{17}+12.4265H_{18}+0.6000 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} -1.6361\delta H_0 -0.0779\delta H_1 +0.2124H_4-0.0128H_{11}-0.0313H_2 -0.6555H_3 -0.2687H_5 +0.2009H_6 \\ -0.2377\ H_7 -0.2099\ H_8 -0.3446H_9+0.5320H_{10}-0.1605H_{12}+0.0096H_{13}-0.1669H_{14}+0.1831H_{15} \\ -0.2723\ H_{16}+1.0125\ H_{17}+1.2273 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} +6.6685\delta H_0 +1.5854\delta H_1+0.0339H_4-0.4925H_{11}+2.0534H_2 +0.0081H_3 +1.4101H_5 +0.1788H_6 \\ +2.3503\ H_7 -0.6104\ H_8+1.1782H_9-0.1682H_{10}+0.3537H_{12}-0.0218H_{13}+0.8939H_{14}+0.0343H_{15} \\ +1.9019\ H_{16}+3.0874 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} -1.0592\delta H_0+0.1190\delta H_1-0.0740H_4-0.0836H_{11}-0.6793H_2 +0.0179H_3 -0.1480H_5 +0.1748H_6 \\ -0.1335\ H_7+0.0422\ H_8-0.4195H_9-0.1693H_{10}-0.1021H_{12}-0.0376H_{13}-0.1846H_{14}+0.3260H_{15} \\ -0.3040 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} +1.0213\delta H_0+0.5177\delta H_1+0.0627H_4-0.1147H_{11}+0.3858H_2 -0.1461H_3 +0.4375H_5 -0.2475H_6 \\ +0.2382\ H_7+0.1854\ H_8+0.1938H_9-0.3222H_{10}+0.1761H_{12}-0.0162H_{13}+0.5471H_{14}+1.2866 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} -0.4299\delta H_0+0.0090\delta H_1+0.1000H_4+0.0100H_{11}+0.0877H_2 +0.0515\ H_3 -0.0900H_5+0.0556H_6 \\ -0.0135\ H_7+0.0506\ H_8+0.0552H_9+0.2606H_{10}-0.0073H_{12}+0.16041H_{13}+0.0441 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} +0.9094\delta H_0+0.3418\delta H_1+0.2536H_4+0.0636H_{11}+0.3663H_2 -0.0444H_3+0.2762H_5+0.0025H_6 \\ +0.0748\ H_7-0.0323\ H_8+0.0002H_9+0.2067H_{10}+0.2230H_{12}-0.4142 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} -3.2817\delta H_0+0.3123\delta H_1-0.2692H_4-0.0468H_{11}+1.9724H_2-1.3333H_3-0.7752H_5-0.4030H_6 \\ +0.5872\ H_7-0.5905\ H_8-1.7876H_9+9.9269H_{10}+6.1521=0, \end{array}\right.$$

$$\left\{\begin{array}{l} -2.1150\delta H_0+0.6245\delta H_1-0.2778H_4-0.2050H_{11}+4.9357H_2-0.1593H_3+0.2598H_5+0.4000H_6 \\ +0.2649\ H_7+0.2398\ H_8+8.1252H_9-0.5478=0, \end{array}\right.$$

$$\left\{\begin{array}{l} -1.5819\delta H_0-0.3390\delta H_1-0.2775H_4+0.0382H_{11}+0.1024H_2-0.6220H_3-0.0385H_5+0.0135H_6 \\ +0.1548\ H_7+1.1412H_8-1.6124=0, \end{array}\right.$$

$$\left\{\begin{array}{l} +0.0847\delta H_0+0.4192\delta H_1-0.0892H_4-0.0086H_{11}+1.1075H_2+0.2911H_3+0.2188H_5+0.0942H_6 \\ +0.6573\ H_7-1.8153=0, \end{array}\right.$$

$$\left\{\begin{array}{l} +1.0952\delta H_0+0.5866\delta H_1-0.0099H_4+0.0179H_{11}-0.2841H_2-0.2528H_3+0.2539H_5+0.3990H_6 \\ +0.3478=0, \end{array}\right.$$

$$-1.2594\delta H_0+0.0841\delta H_1-0.0889H_4-0.0249H_{11}-0.1172H_2-0.0961H_3+0.1979H_5+1.2564=0,$$

$$+9.7240\delta H_0-1.7733\delta H_1-0.2521H_4+0.1667H_{11}+0.7938H_2+8.0256H_3-0.8292=0,$$

$$+1.6929\delta H_0+0.0284\delta H_1+0.0318H_4-0.3858H_{11}+4.6806H_2-3.4416=0,$$

$$+0.3130\delta H_0+0.3043\delta H_1+0.1724H_4+0.1658H_{11}+0.0313=0,$$

$$-0.0319\delta H_0+0.7090\delta H_1+0.4432H_4+1.5519=0,$$

$$+0.4345\delta H_0+1.0385\delta H_1+1.1963=0,$$

$$+8.1849\delta H_0-1.1217=0.$$

The values of the several constants are

$\delta H_0=+0.137,$	$H_5=-4.918,$	$H_{10}=-1.344,$	$H_{15}=+2.476,$
$\delta H_1=-1.2095,$	$H_6=+3.822,$	$H_{11}=+3.391,$	$H_{16}=-3.057,$
$H_2=+0.984,$	$H_7=+3.023,$	$H_{12}=+8.474,$	$H_{17}=-1.886,$
$H_3=-0.547,$	$H_8=-0.255,$	$H_{13}=-0.450,$	$H_{18}=-0.248,$
$H_4=-1.557,$	$H_9=-0.500,$	$H_{14}=-0.336,$	$H_{19}=+0.624.$

The sum of the squares of the residuals is diminished from 65.859 to 18.690.

Applying the corrections to the adopted approximate values of H_0 and H_1, we have

$$H_0=\frac{c^2}{M}=11458.729, \qquad H_1=-20.0680.$$

A sufficiently approximate relation between c and M is

$$c=M-\frac{1}{3}H_1+\frac{1}{35}H_{11}+2\pi^2;$$

which gives

$$\frac{(M+26.5263)^2}{M}=11458.729;$$

whence

$$M=11405.615.$$

Thus, as the result of the discussion, we have

$$\frac{3}{2}\frac{C-\frac{A+B}{2}}{MD^2} = -\frac{H_1}{M} = 0.001759484.$$

Although it is unnecessary for our purpose, the resulting expression for L, the length of the second's pendulum, may be given. It is in meters, and it must be understood that the unit of r is the average of all the equatorial radii.*

$$\begin{aligned}
L = \;& \overset{m.}{0.9927148} \\
&+ 0.0050890\, r^{-4}\left(\sin^2\varphi - \frac{1}{3}\right) \\
&+ 0.0000979\, r^{-4}\cos^2\varphi\cos(2\omega' + 29^\circ\, 4') \\
&- 0.0001355\, r^{-5}\left(\sin^3\varphi - \frac{3}{5}\sin\varphi\right) \\
&+ 0.0005421\, r^{-5}\left(\sin^2\varphi - \frac{1}{5}\right)\cos\varphi\cos(\omega' + 217^\circ\, 51') \\
&+ 0.0002640\, r^{-5}\sin\varphi\cos^2\varphi\cos(2\omega' + 4^\circ\, 49') \\
&+ 0.0001248\, r^{-5}\cos^3\varphi\cos(3\omega' + 110^\circ\, 24') \\
&+ 0.0001489\, r^{-6}\left(\sin^4\varphi - \frac{6}{7}\sin^2\varphi + \frac{3}{35}\right) \\
&+ 0.0007386\, r^{-6}\left(\sin^3\varphi - \frac{3}{7}\sin\varphi\right)\cos\varphi\cos(\omega' + 3^\circ\, 2') \\
&+ 0.0002175\, r^{-6}\left(\sin^2\varphi - \frac{1}{7}\right)\cos^2\varphi\cos(2\omega' + 262^\circ\, 17') \\
&+ 0.0003126\, r^{-6}\sin\varphi\cos^3\varphi\cos(3\omega' + 148^\circ\, 20') \\
&+ 0.0000584\, r^{-6}\cos^4\varphi\cos(4\omega' + 248^\circ\, 19').
\end{aligned}$$

The relative importance of the several terms of this formula is exhibited in the following table, which gives half the range of value of each variable term:

2d term 0.0025445,	8th term 0.0000137,
3d term 0.0000979,	9th term 0.0001114,
4th term 0.0000542,	10th term 0.0000400,
5th term 0.0001493,	11th term 0.0001015,
6th term 0.0001016,	12th term 0.0000584.
7th term 0.0001248,	

The observed values, given above, are represented by this formula with residuals which have been given with the observations themselves.

*All these formulæ have been corrected for the oversight mentioned in a preceding note. The mean compressions, derived from them for the Northern and Southern Hemispheres, are, respectively, $\frac{1}{285.44}$ and $\frac{1}{290.02}$.

CHAPTER VI.

NUMERICAL EXPRESSIONS FOR THE PERTURBATIONS OF THE CO-ORDINATES OF THE MOON PRODUCED BY THE FIGURE OF THE EARTH.

The value of the principal factor, which has been obtained in the preceding chapter, being substituted in the expressions for β_1, β_2, and β_3, given in Chapter I, and the mean obliquity of the ecliptic at the epoch 1850.0 being taken as

$$\varepsilon = 23^\circ\ 27'\ 31''.84,$$

we get, in seconds of arc,

$$\frac{\beta_1}{a^2} = 0''.07603735, \qquad \frac{\beta_2}{a^2} = 0''.07285405, \qquad \frac{\beta_3}{a^2} = 0''.01580782.$$

And the longitude of the solar perigee at the epoch 1850.0 is (HANSEN *et* OLUFSEN, *Tables du Soleil*, p. 1),

$$\psi + h' + g' = 280^\circ\ 21'\ 41''.$$

The remaining quantities which we need for the reduction of the coefficients to numbers will be taken from DELAUNAY (*Théorie du Mouvement de la Lune*, Tom. II, pp. 801–803). They are

$$m = 0.07480133, \qquad e = 0.0548993,$$
$$\gamma = 0.04488663, \qquad e' = 0.01677106,$$
$$a = 60.31854, \qquad \frac{a}{a'} = 0.00255878,$$
$$\frac{f}{n} = 0.000002908, \qquad n = 17325594''.$$

When these values are substituted in the expressions of Chapter IV, we obtain:

The Value of δV.			
	$''$		$''$
1	$-\ 0.0006 \sin l'$	7	$+\ 0.0210 \sin (2D - l)$
2	$-\ 0.0002 \sin 2l$	8	$+\ 0.0004 \sin (2D - l - l')$
3	$+\ 0.0008 \sin 2F$	9	$-\ 0.0005 \sin (2D - l + l')$
4	$+\ 0.0041 \sin (2F - l)$	10	$+\ 0.0001 \sin (2D - l + 2l')$
5	$-\ 0.0015 \sin (2F - 2l)$	11	$-\ 0.0009 \sin (2D - 2l)$
6	$-\ 0.0009 \sin 2D$	12	$+\ 0.0005 \sin (2D - 2F)$

The Value of δV—Continued.

	$''$		$''$
13	$+ 0.0014 \sin (2D - 2F + l)$	62	$+ 0.0010 \sin (\zeta + 2D + F - 2l)$
14	$- 0.0004 \sin D$	63	$+ 0.0961 \sin (\zeta + 2D - F)$
15	$+ 0.0001 \sin (D + l')$	64	$+ 0.0040 \sin (\zeta + 2D - F - l')$
16	$- 0.0001 \sin (D + l + l')$	65	$0.0000 \sin (\zeta + 2D - F - 2l')$
17	$- 0.0003 \sin (D - l')$	66	$- 0.0008 \sin (\zeta + 2D - F + l')$
18	$+ 0.0004 \sin (D - l + l')$	67	$0.0000 \sin (\zeta + 2D - F + 2l')$
19	$+ 0.3908 \sin (\zeta + F)$	68	$+ 0.0089 \sin (\zeta + 2D - F + l)$
20	$+ 0.0003 \sin (\zeta + F - l')$	69	$0.0000 \sin (\zeta + 2D - F + l - l')$
21	$0.0000 \sin (\zeta + F - 2l')$	70	$0.0000 \sin (\zeta + 2D - F + l + l')$
22	$- 0.0006 \sin (\zeta + F + l')$	71	$+ 0.0001 \sin (\zeta + 2D - F + 2l)$
23	$0.0000 \sin (\zeta + F + 2l')$	72	$+ 0.0713 \sin (\zeta + 2D - F - l)$
24	$+ 0.0420 \sin (\zeta + F + l)$	73	$+ 0.0023 \sin (\zeta + 2D - F - l - l')$
25	$+ 0.0002 \sin (\zeta + F + l - l')$	74	$- 0.0010 \sin (\zeta + 2D - F - l + l')$
26	$- 0.0002 \sin (\zeta + F + l + l')$	75	$- 0.0002 \sin (\zeta + 2D - F - 2l)$
27	$+ 0.0039 \sin (\zeta + F + 2l)$	76	$0.0000 \sin (\zeta + 2D - 3F)$
28	$+ 0.0003 \sin (\zeta + F + 3l)$	77	$+ 0.0520 \sin (\zeta - 2D + F)$
29	$+ 0.0551 \sin (\zeta + F - l)$	78	$- 0.0014 \sin (\zeta - 2D + F - l')$
30	$0.0000 \sin (\zeta + F - l - l')$	79	$0.0000 \sin (\zeta - 2D + F - 2l')$
31	$0.0000 \sin (\zeta + F - l + l')$	80	$+ 0.0022 \sin (\zeta - 2D + F + l')$
32	$- 0.0035 \sin (\zeta + F - 2l)$	81	$+ 0.0001 \sin (\zeta - 2D + F + 2l')$
33	$- 0.0003 \sin (\zeta + F - 3l)$	82	$+ 0.0008 \sin (\zeta - 2D + F + l)$
34	$- 0.0008 \sin (\zeta + 3F)$	83	$0.0000 \sin (\zeta - 2D + F + l - l')$
35	$- 0.0002 \sin (\zeta + 3F + l)$	84	$+ 0.0001 \sin (\zeta - 2D + F + l + l')$
36	$- 0.0003 \sin (\zeta + 3F - l)$	85	$- 0.0002 \sin (\zeta - 2D + F + 2l)$
37	$+ 7.6708 \sin (\zeta - F)$	86	$- 0.0093 \sin (\zeta - 2D + F - l)$
38	$+ 0.0033 \sin (\zeta - F - l')$	87	$+ 0.0002 \sin (\zeta - 2D + F - l - l')$
39	$0.0000 \sin (\zeta - F - 2l')$	88	$- 0.0005 \sin (\zeta - 2D + F - l + l')$
40	$- 0.0029 \sin (\zeta - F + l')$	89	$- 0.0008 \sin (\zeta - 2D + F - 2l)$
41	$0.0000 \sin (\zeta - F + 2l')$	90	$- 0.0002 \sin (\zeta - 2D + 3F)$
42	$+ 0.5199 \sin (\zeta - F + l)$	91	$+ 0.0642 \sin (\zeta - 2D - F)$
43	$+ 0.0018 \sin (\zeta - F + l - l')$	92	$- 0.0002 \sin (\zeta - 2D - F - l')$
44	$- 0.0018 \sin (\zeta - F + l + l')$	93	$0.0000 \sin (\zeta - 2D - F - 2l')$
45	$+ 0.0343 \sin (\zeta - F + 2l)$	94	$+ 0.0027 \sin (\zeta - 2D - F + l')$
46	$+ 0.0022 \sin (\zeta - F + 3l)$	95	$0.0000 \sin (\zeta - 2D - F + 2l')$
47	$+ 0.5193 \sin (\zeta - F - l)$	96	$+ 0.0584 \sin (\zeta - 2D - F + l)$
48	$- 0.0010 \sin (\zeta - F - l - l')$	97	$- 0.0008 \sin (\zeta - 2D - F + l - l')$
49	$+ 0.0010 \sin (\zeta - F - l + l')$	98	$+ 0.0019 \sin (\zeta - 2D - F + l + l')$
50	$+ 0.0331 \sin (\zeta - F - 2l)$	99	$- 0.0003 \sin (\zeta - 2D - F + 2l)$
51	$+ 0.0020 \sin (\zeta - F - 3l)$	100	$+ 0.0058 \sin (\zeta - 2D - F - l)$
52	$+ 0.0160 \sin (\zeta - 3F)$	101	$0.0000 \sin (\zeta - 2D - F - l - l')$
53	$- 0.0011 \sin (\zeta - 3F + l)$	102	$0.0000 \sin (\zeta - 2D - F - l + l')$
54	$- 0.0026 \sin (\zeta - 3F - l)$	103	$- 0.0001 \sin (\zeta - 2D - F - 2l)$
55	$+ 0.0049 \sin (\zeta + 2D + F)$	104	$0.0000 \sin (\zeta - 2D - 3F)$
56	$+ 0.0002 \sin (\zeta + 2D + F - l')$	105	$+ 0.0001 \sin (\zeta + 4D - F)$
57	$0.0000 \sin (\zeta + 2D + F + l')$	106	$+ 0.0002 \sin (\zeta + 4D - F - l)$
58	$+ 0.0006 \sin (\zeta + 2D + F + l)$	107	$0.0000 \sin (\zeta - 4D + F)$
59	$+ 0.0087 \sin (\zeta + 2D + F - l)$	108	$0.0000 \sin (\zeta - 4D + F - l')$
60	$+ 0.0002 \sin (\zeta + 2D + F - l - l')$	109	$0.0000 \sin (\zeta - 4D + F + l')$
61	$- 0.0001 \sin (\zeta + 2D + F - l + l')$	110	$+ 0.0004 \sin (\zeta - 4D + F + l)$

The Value of δV—Continued.

	$''$		$''$
111	$0.0000 \sin(\zeta - 4D + F - l)$	139	$0.0000 \sin(2\zeta - 2F + l')$
112	$-0.0001 \sin(\zeta - 4D - F)$	140	$-0.0025 \sin(2\zeta - 2F + l)$
113	$-0.0002 \sin(\zeta - 4D - F + l)$	141	$-0.0002 \sin(2\zeta - 2F + 2l)$
114	$-0.0001 \sin(\zeta + D + F)$	142	$-0.0025 \sin(2\zeta - 2F - l)$
115	$0.0000 \sin(\zeta + D + F + l')$	143	$-0.0001 \sin(2\zeta - 2F - 2l)$
116	$0.0000 \sin(\zeta + D + F - l + l')$	144	$+0.0001 \sin(2\zeta - 4F)$
117	$-0.0021 \sin(\zeta + D - F)$	145	$-0.0002 \sin(2\zeta + 2D)$
118	$+0.0007 \sin(\zeta + D - F + l')$	146	$-0.0001 \sin(2\zeta + 2D - l)$
119	$0.0000 \sin(\zeta + D + F + l + l')$	147	$-0.0004 \sin(2\zeta + 2D - 2F)$
120	$-0.0001 \sin(\zeta + D - F - l + l')$	148	$0.0000 \sin(2\zeta + 2D - 2F - l')$
121	$0.0000 \sin(\zeta + D - F - 2l + l')$	149	$0.0000 \sin(2\zeta + 2D - 2F + l')$
122	$-0.0007 \sin(\zeta - D + F)$	150	$0.0000 \sin(2\zeta + 2D - 2F + l)$
123	$0.0000 \sin(\zeta - D + F - l')$	151	$-0.0003 \sin(2\zeta + 2D - 2F - l)$
124	$-0.0006 \sin(\zeta - D - F)$	152	$-0.0005 \sin(2\zeta - 2D)$
125	$+0.0003 \sin(\zeta - D - F - l')$	153	$0.0000 \sin(2\zeta - 2D - l')$
126	$0.0000 \sin(\zeta - D - F + l')$	154	$0.0000 \sin(2\zeta - 2D + l')$
127	$-0.0002 \sin(\zeta - 3D + F)$	155	$0.0000 \sin(2\zeta - 2D + l)$
128	$-0.0025 \sin 2\zeta$	156	$+0.0002 \sin(2\zeta - 2D - l)$
129	$0.0000 \sin(2\zeta - l')$	157	$0.0000 \sin(2\zeta - 2D + 2F)$
130	$0.0000 \sin(2\zeta + l')$	158	$-0.0002 \sin(2\zeta - 2D - 2F)$
131	$-0.0005 \sin(2\zeta + l)$	159	$0.0000 \sin(2\zeta - 2D - 2F - l')$
132	$0.0000 \sin(2\zeta + 2l)$	160	$0.0000 \sin(2\zeta - 2D - 2F + l')$
133	$+0.0007 \sin(2\zeta - l)$	161	$-0.0002 \sin(2\zeta - 2D - 2F + l)$
134	$+0.0002 \sin(2\zeta - 2l)$	162	$0.0000 \sin(2\zeta - 2D - 2F - l)$
135	$0.0000 \sin(2\zeta + 2F)$	163	$0.0000 \sin(2\zeta - 4D)$
136	$0.0000 \sin(2\zeta + 2F - 2l)$	164	$+0.0002 \sin(2\zeta - 4D + l)$
137	$-0.0395 \sin(2\zeta - 2F)$	165	$+0.0002 \sin(2\zeta - D - l')$
138	$0.0000 \sin(2\zeta - 2F - l')$		

The Value of δU.

	$''$		$''$
1	$+0.0005 \sin(F + l)$	17	$-0.0035 \sin(\zeta - l')$
2	$-0.0005 \sin(F - l)$	18	$-0.0001 \sin(\zeta - 2l')$
3	$+0.0013 \sin(F - 2l)$	19	$0.0000 \sin(\zeta - 3l')$
4	$+0.0002 \sin(F - 3l)$	20	$+0.0029 \sin(\zeta + l')$
5	$+0.0004 \sin(3F - l)$	21	$+0.0001 \sin(\zeta + 2l')$
6	$+0.0007 \sin(2D + F - l)$	22	$0.0000 \sin(\zeta + 3l')$
7	$-0.0001 \sin(2D + F - 2l)$	23	$-0.4533 \sin(\zeta + l)$
8	$-0.0025 \sin(2D - F)$	24	$-0.0027 \sin(\zeta + l - l')$
9	$-0.0001 \sin(2D - F - l')$	25	$0.0000 \sin(\zeta + l - 2l')$
10	$+0.0001 \sin(2D - F + l')$	26	$+0.0024 \sin(\zeta + l + l')$
11	$0.0000 \sin(2D - F + 2l')$	27	$0.0000 \sin(\zeta + l + 2l')$
12	$-0.0001 \sin(2D - F + l)$	28	$-0.0196 \sin(\zeta + 2l)$
13	$+0.0008 \sin(2D - F - l)$	29	$-0.0002 \sin(\zeta + 2l - l')$
14	$-0.0001 \sin(D + F + l')$	30	$+0.0002 \sin(\zeta + 2l + l')$
15	$-0.0001 \sin(D - F + l')$	31	$-0.0020 \sin(\zeta + 3l)$
16	$-8.7256 \sin \zeta$	32	$-0.0001 \sin(\zeta + 4l)$

The Value of δU—Continued.

	$''$		$''$
33	$+ 0.4930 \sin (\zeta - l)$	83	$+ 0.3228 \sin (\zeta - 2D)$
34	$- 0.0020 \sin (\zeta - l - l')$	84	$- 0.0062 \sin (\zeta - 2D - l')$
35	$0.0000 \sin (\zeta - l - 2l')$	85	$- 0.0001 \sin (\zeta - 2D - 2l')$
36	$+ 0.0020 \sin (\zeta - l + l')$	86	$0.0000 \sin (\zeta - 2D - 3l')$
37	$0.0000 \sin (\zeta - l + 2l')$	87	$+ 0.0148 \sin (\zeta - 2D + l')$
38	$+ 0.0193 \sin (\zeta - 2l)$	88	$+ 0.0005 \sin (\zeta - 2D + 2l')$
39	$- 0.0001 \sin (\zeta - 2l - l')$	89	$+ 0.0782 \sin (\zeta - 2D + l)$
40	$+ 0.0001 \sin (\zeta - 2l + l')$	90	$- 0.0010 \sin (\zeta - 2D + l - l')$
41	$+ 0.0009 \sin (\zeta - 3l)$	91	$0.0000 \sin (\zeta - 2D + l - 2l')$
42	$+ 0.0001 \sin (\zeta - 4l)$	92	$+ 0.0031 \sin (\zeta - 2D + l + l')$
43	$+ 0.0092 \sin (\zeta + 2F)$	93	$+ 0.0001 \sin (\zeta - 2D + l + 2l')$
44	$0.0000 \sin (\zeta + 2F - l')$	94	$+ 0.0066 \sin (\zeta - 2D + 2l)$
45	$0.0000 \sin (\zeta + 2F + l')$	95	$- 0.0001 \sin (\zeta - 2D + 2l - l')$
46	$+ 0.0014 \sin (\zeta + 2F + l)$	96	$+ 0.0002 \sin (\zeta - 2D + 2l + l')$
47	$+ 0.0002 \sin (\zeta + 2F + 2l)$	97	$+ 0.0004 \sin (\zeta - 2D + 3l)$
48	$+ 0.0046 \sin (\zeta + 2F - l)$	98	$+ 0.0175 \sin (\zeta - 2D - l)$
49	$- 0.0004 \sin (\zeta + 2F - 2l)$	99	$- 0.0003 \sin (\zeta - 2D - l - l')$
50	$0.0000 \sin (\zeta + 4F)$	100	$0.0000 \sin (\zeta - 2D - l - 2l')$
51	$+ 0.3523 \sin (\zeta - 2F)$	101	$+ 0.0007 \sin (\zeta - 2D - l + l')$
52	$- 0.0001 \sin (\zeta - 2F - l')$	102	$0.0000 \sin (\zeta - 2D - l + 2l')$
53	$+ 0.0001 \sin (\zeta - 2F + l')$	103	$+ 0.0010 \sin (\zeta - 2D - 2l)$
54	$+ 0.0011 \sin (\zeta - 2F + l)$	104	$0.0000 \sin (\zeta - 2D - 2l - l')$
55	$+ 0.0008 \sin (\zeta - 2F + 2l)$	105	$0.0000 \sin (\zeta - 2D - 2l + l')$
56	$+ 0.0411 \sin (\zeta - 2F - l)$	106	$+ 0.0001 \sin (\zeta - 2D - 3l)$
57	$+ 0.0035 \sin (\zeta - 2F - 2l)$	107	$+ 0.0032 \sin (\zeta - 2D + 2F)$
58	$- 0.0003 \sin (\zeta - 4F)$	108	$- 0.0001 \sin (\zeta - 2D + 2F - l')$
59	$- 0.0515 \sin (\zeta + 2D)$	109	$+ 0.0001 \sin (\zeta - 2D + 2F + l')$
60	$- 0.0033 \sin (\zeta + 2D - l')$	110	$+ 0.0006 \sin (\zeta - 2D + 2F + l)$
61	$- 0.0001 \sin (\zeta + 2D - 2l')$	111	$- 0.0006 \sin (\zeta - 2D + 2F - l)$
62	$+ 0.0006 \sin (\zeta + 2D + l')$	112	$+ 0.0035 \sin (\zeta - 2D - 2F)$
63	$- 0.0067 \sin (\zeta + 2D + l)$	113	$0.0000 \sin (\zeta - 2D - 2F - l')$
64	$- 0.0003 \sin (\zeta + 2D + l - l')$	114	$0.0000 \sin (\zeta - 2D + 2F + l')$
65	$0.0000 \sin (\zeta + 2D + l + l')$	115	$- 0.0048 \sin (\zeta - 2D - 2F + l)$
66	$- 0.0005 \sin (\zeta + 2D + 2l)$	116	$0.0000 \sin (\zeta - 2D - 2F - l)$
67	$- 0.0898 \sin (\zeta + 2D - l)$	117	$- 0.0002 \sin (\zeta + 4D)$
68	$- 0.0039 \sin (\zeta + 2D - l - l')$	118	$- 0.0007 \sin (\zeta + 4D - l)$
69	$- 0.0001 \sin (\zeta + 2D - l - 2l')$	119	$- 0.0006 \sin (\zeta + 4D - 2l)$
70	$+ 0.0013 \sin (\zeta + 2D - l + l')$	120	$0.0000 \sin (\zeta + 4D - 2F)$
71	$0.0000 \sin (\zeta + 2D - l + 2l')$	121	$+ 0.0015 \sin (\zeta - 4D)$
72	$+ 0.0006 \sin (\zeta + 2D - 2l)$	122	$0.0000 \sin (\zeta - 4D - l')$
73	$0.0000 \sin (\zeta + 2D - 2l - l')$	123	$+ 0.0001 \sin (\zeta - 4D + l')$
74	$0.0000 \sin (\zeta + 2D - 2l + l')$	124	$+ 0.0028 \sin (\zeta - 4D + l)$
75	$- 0.0001 \sin (\zeta + 2D - 3l)$	125	$- 0.0001 \sin (\zeta - 4D + l - l')$
76	$+ 0.0001 \sin (\zeta + 2D + 2F)$	126	$+ 0.0001 \sin (\zeta - 4D + l + l')$
77	$+ 0.0002 \sin (\zeta + 2D + 2F - l)$	127	$+ 0.0002 \sin (\zeta - 4D + 2l)$
78	$+ 0.0102 \sin (\zeta + 2D - 2F)$	128	$+ 0.0001 \sin (\zeta - 4D - l)$
79	$+ 0.0003 \sin (\zeta + 2D - 2F - l')$	129	$+ 0.0004 \sin (\zeta - 4D + 2F)$
80	$- 0.0001 \sin (\zeta + 2D - 2F + l')$	130	$+ 0.0023 \sin (\zeta + D)$
81	$+ 0.0011 \sin (\zeta + 2D - 2F + l)$	131	$0.0000 \sin (\zeta + D - l')$
82	$0.0000 \sin (\zeta + 2D - 2F - l)$	132	$- 0.0004 \sin (\zeta + D + l')$

The Value of δU—Continued.

	$''$		$''$
133	$+ 0.0002 \sin (\zeta + D + l)$	172	$- 0.0016 \sin (2\zeta - 3F)$
134	$- 0.0001 \sin (\zeta + D + l + l')$	173	$0.0000 \sin (2\zeta - 3F + l)$
135	$- 0.0001 \sin (\zeta + D - l)$	174	$- 0.0002 \sin (2\zeta - 3F - l)$
136	$+ 0.0001 \sin (\zeta + D - l + l')$	175	$+ 0.0005 \sin (2\zeta + 2D - F)$
137	$0.0000 \sin (\zeta + D - 2l + l')$	176	$0.0000 \sin (2\zeta + 2D - F - l')$
138	$0.0000 \sin (\zeta + D - 2F + l')$	177	$0.0000 \sin (2\zeta + 2D - F + l')$
139	$0.0000 \sin (\zeta + D - 2F - l + l')$	178	$0.0000 \sin (2\zeta + 2D - F + l)$
140	$- 0.0020 \sin (\zeta - D)$	179	$+ 0.0009 \sin (2\zeta + 2D - F - l)$
141	$+ 0.0004 \sin (\zeta - D - l')$	180	$0.0000 \sin (2\zeta + 2D - F - l - l')$
142	$0.0000 \sin (\zeta - D + l')$	181	$0.0000 \sin (2\zeta + 2D - F - l + l')$
143	$- 0.0001 \sin (\zeta - D + l)$	182	$0.0000 \sin (2\zeta + 2D - 3F)$
144	$0.0000 \sin (\zeta - D + l - l')$	183	$- 0.0001 \sin (2\zeta - 2D + F)$
145	$- 0.0003 \sin (\zeta - D - l)$	184	$0.0000 \sin (2\zeta - 2D + F - l')$
146	$0.0000 \sin (\zeta - D - l - l')$	185	$0.0000 \sin (2\zeta - 2D + F + l')$
147	$0.0000 \sin (\zeta - D - l + l')$	186	$0.0000 \sin (2\zeta - 2D + F + l)$
148	$0.0000 \sin (\zeta + 3D)$	187	$0.0000 \sin (2\zeta - 2D + F - l)$
149	$- 0.0001 \sin (\zeta - 3D)$	188	$0.0000 \sin (2\zeta - 2D + F - 2l)$
150	$0.0000 \sin (\zeta - 3D - l')$	189	$- 0.0032 \sin (2\zeta - 2D - F)$
151	$- 0.0001 \sin (\zeta - 3D + l)$	190	$0.0000 \sin (2\zeta - 2D - F - l')$
152	$0.0000 \sin (2\zeta + F)$	191	$0.0000 \sin (2\zeta - 2D - F - 2l')$
153	$0.0000 \sin (2\zeta + F + l)$	192	$- 0.0001 \sin (2\zeta - 2D - F + l')$
154	$- 0.0001 \sin (2\zeta + F - l)$	193	$0.0000 \sin (2\zeta - 2D - F + 2l')$
155	$0.0000 \sin (2\zeta + F - 2l)$	194	$- 0.0007 \sin (2\zeta - 2D - F + l)$
156	$0.0000 \sin (2\zeta + F - 3l)$	195	$0.0000 \sin (2\zeta - 2D - F + l - l')$
157	$+ 0.0873 \sin (2\zeta - F)$	196	$0.0000 \sin (2\zeta - 2D - F + l + l')$
158	$0.0000 \sin (2\zeta - F - l')$	197	$0.0000 \sin (2\zeta - 2D - F + 2l)$
159	$0.0000 \sin (2\zeta - F - 2l')$	198	$- 0.0001 \sin (2\zeta - 2D - F - l)$
160	$0.0000 \sin (2\zeta - F + l')$	199	$0.0000 \sin (2\zeta - 2D - F - l - l')$
161	$0.0000 \sin (2\zeta - F + 2l')$	200	$0.0000 \sin (2\zeta - 2D - F - l + l')$
162	$+ 0.0046 \sin (2\zeta - F + l)$	201	$0.0000 \sin (2\zeta - 2D - F - 2l)$
163	$0.0000 \sin (2\zeta - F + l - l')$	202	$0.0000 \sin (2\zeta - 2D - 3F)$
164	$0.0000 \sin (2\zeta - F + l + l')$	203	$0.0000 \sin (2\zeta - 4D + F)$
165	$+ 0.0003 \sin (2\zeta - F + 2l)$	204	$0.0000 \sin (2\zeta - 4D - F)$
166	$0.0000 \sin (2\zeta - F + 3l)$	205	$0.0000 \sin (2\zeta - 4D - F + l)$
167	$- 0.0048 \sin (2\zeta - F - l)$	206	$0.0000 \sin (2\zeta + D - F)$
168	$0.0000 \sin (2\zeta - F - l - l')$	207	$0.0000 \sin (2\zeta + D - F + l')$
169	$0.0000 \sin (2\zeta - F - l + l')$	208	$0.0000 \sin (2\zeta - D - F)$
170	$- 0.0002 \sin (2\zeta - F - 2l)$	209	$0.0000 \sin (2\zeta - D - F - l')$
171	$0.0000 \sin (2\zeta - F - 3l)$		

The Value of $\delta \frac{1}{r}$.

	$''$		$''$
1	$- 0.0004$	4	$- 0.0035 \cos (\zeta - F - l)$
2	$+ 0.0012 \cos (\zeta + F - l)$	5	$0.0000 \cos 2\zeta$
3	$+ 0.0035 \cos (\zeta - F + l)$		

The motions of the perigee and node are, the unit of time being the Julian year,

$$\frac{d(g+h)}{dt} = + 6''.7725, \qquad \frac{dh}{dt} = - 6''.4128.$$

MEMOIR No. 49

ON CERTAIN LUNAR INEQUALITIES DUE TO THE ACTION OF JUPITER AND DISCOVERED BY MR. E. NEISON

(Astronomical Papers of the American Ephemeris, Vol. III, pp. 373-393, 1885.)

ON CERTAIN LUNAR INEQUALITIES DUE TO THE ACTION OF JUPITER, AND DISCOVERED BY MR. E. NEISON.

About ten years ago Professor NEWCOMB, in discussing the corrections which the observations of the moon indicated to the Nautical Almanac values of the longitude, was led to advocate the existence of a new inequality, with a coefficient of $1''.5$ in the longitude, and having a period of about seventeen years as regards its effect on the eccentricity and longitude of the perigee.

A short time after the publication of this, Mr. E. NEISON was so fortunate as to find in the action of Jupiter the explanation of this inequality. In two short notes communicated to the Royal Astronomical Society,* the latter being written mainly for the purpose of correcting the former, Mr. NEISON gives the final numerical results of his investigation, with a statement of the great labor and difficulty involved in their production, but without any detail as to the intermediate steps.

Using DELAUNAY's notation for arguments, Mr. NEISON's expression for the inequalities in longitude is

$$\delta V = -1''.163 \sin(2h + 2g + l - 2h'' - 2g'' - 2l'') + 2''.200 \sin(2h + 2g - 2h'' - 2g'' - 2l'')$$

It will be noticed that in the latter term of this, Mr. NEISON has the associated long period inequality in the mean longitude, which it would not have been possible for Professor NEWCOMB to have elicited from his discussion on account of the near approach of its period to that of a revolution of the moon's node.

Although eight years have elapsed since the publication of these two notes, their author has not yet given us the analysis which led him to these inequalities. And, so far as I know, no one else has published anything in relation to the matter. Still these terms are interesting as being the only sensible ones which have been thus far detected from the action of Jupiter. Moreover, the coefficient of the second of the inequalities mentioned above is, by theory, a quantity one order higher than that of the first; the first having the simple power of the eccentricity as factor, while the second has the square. Hence we should naturally expect to find the latter coefficient the smaller. Thus there arises in one's mind the suspicion that Mr. NEISON's value is too large.

In the discussion which follows I propose to determine the coefficients of these inequalities to such a degree of exactitude that the highest order of terms taken into account shall exceed by two orders the lowest order appearing in the coefficients. Thus, in general, three orders of terms will be present in the coefficients. To this extent it is found that about ten days' work suffice for the elaboration. The method

* Monthly Notices, Vol. XXXVII, pp. 248, 358.

used is that of Delaunay, which in this class of inequalities appears to me to be far superior to any other that has been imagined

We have here to consider both the direct and indirect action of the planet, but the latter is of quite inferior importance. Hence we attend to the direct action first.

I.—Terms of the Perturbative Function Arising from the Direct Action of Jupiter.

In determining the lunar perturbations which arise from the direct action of a planet it generally suffices to reduce R to the following expression:*

$$R = \frac{m''}{m'} m' \frac{a^2}{a'^3} \left\{ \frac{1}{4} \left[\frac{a'^3}{\Delta^3} - 3 \frac{a'^3 z''^2}{\Delta^5} \right] \frac{r^2 - 3z^2}{a^2} + \frac{3}{4} a'^3 \frac{(x''+x')^2 - (y''+y')^2}{\Delta^5} \frac{x^2 - y^2}{a^2} \right.$$
$$\left. + 3a'^3 \frac{(x''+x')(y''+y')}{\Delta^5} \frac{xy}{a^2} + 3a'^3 \frac{(x''+x')z''}{\Delta^5} \frac{xz}{a^2} + 3a'^3 \frac{(y''+y')z''}{\Delta^5} \frac{yz}{a^2} \right\}$$

Here the geocentric co-ordinates of the moon are denoted by symbols without accents, those of the sun by symbols with one accent, and the heliocentric co-ordinates of Jupiter by two accents. The two last terms of this expression, having z as a factor, when developed in periodic series, give rise to terms having an odd multiple of h in their arguments; consequently we do not need to consider them. Also in the first term the portions having z''^2 or z^2 as a factor, have, in the terms we need to consider, besides the factors γ''^2 or γ^2, some power of $\frac{n'}{n}$ as a factor, and, in consequence, are of higher orders than we propose to retain. Thus we may write

$$R = \frac{m''}{m'} m' \frac{a^2}{a'^3} \left\{ \frac{1}{4} \frac{a'^3}{\Delta^3} \frac{r^2}{a^2} + \frac{3}{4} \frac{a'^3}{\Delta^5} \left[(x''+x')^2 - (y''+y')^2 \right] \frac{x^2 - y^2}{a^2} + 3 \frac{a'^3}{\Delta^5} (x''+x')(y''+y') \frac{xy}{a^2} \right\}$$

Attending first to the development of this when elliptic values are attributed to the moon's co-ordinates, it will be sufficient in the first term to put

$$\frac{r^2}{a^2} = 1 + \frac{3}{2} e^2 - \left(2e - \frac{e^3}{4} \right) \cos l - \frac{1}{2} e^2 \cos 2l - \frac{1}{4} e^3 \cos 3l$$

and

$$\frac{a'^3}{\Delta^3} = \alpha^3 b_{\frac{3}{2}}^{(2)} \cos (2h' + 2g' + 2l' - 2h'' - 2g'' - 2l'')$$

In the remaining terms of R we substitute, the notation being that of Delaunay,

$$x' = r' \cos (\nu' + h')$$
$$y' = r' \sin (\nu' + h')$$
$$x'' = (1 - \gamma''^2) r'' \cos (\nu'' + h'') + \gamma''^2 r'' \cos (\nu'' - h'')$$
$$y'' = (1 - \gamma''^2) r'' \sin (\nu'' + h'') - \gamma''^2 r'' \sin (\nu'' - h'')$$
$$\Delta^2 = (x''+x')^2 + (y''+y')^2 + z''^2 = r''^2 + 2r''r'S + r'^2$$
$$S = (1 - \gamma''^2) \cos (\nu'' + h'' - \nu' - h') + \gamma''^2 \cos (\nu'' - h'' + \nu' + h')$$

But the second terms of x'', y'', and S have no influence on the terms we seek, hence it is allowable to put

* See American Journal of Mathematics, Vol. VI, p. 115.

$$x''^2 - y''^2 = (1 - \gamma''^2)^2 r''^2 \cos 2(v'' + h'')$$
$$2x''y'' = (1 - \gamma''^2)^2 r''^2 \sin 2(v'' + h'')$$
$$x''x' - y''y' = (1 - \gamma''^2) r''r' \cos(v'' + h'' + v' + h')$$
$$x''y' + y''x' = (1 - \gamma''^2) r''r' \sin(v'' + h'' + v' + h')$$
$$\triangle^2 = r''^2 + 2(1 - \gamma''^2) r''r' \cos(v'' + h'' - v' - h') + r'^2$$

In like manner it will suffice for our purpose to put

$$\frac{x^2 - y^2}{a^2} = (1 - \gamma^2)^2 \Sigma . \mathrm{H}^{(i)} \cos(2h + 2g + il)$$
$$2\frac{xy}{a^2} = (1 - \gamma^2)^2 \Sigma . \mathrm{H}^{(i)} \sin(2h + 2g + il)$$

where the summation must be extended to all integral values of i, both positive and negative, and where

$$\mathrm{H}^{(i)} = \frac{2}{i}\left[\left(\cos^2\frac{\varphi}{2} - \frac{1}{4}e^2\right)\mathrm{J}_{ie}^{(i-2)} - e\cos^2\frac{\varphi}{2}.\ \mathrm{J}_{ie}^{(i-1)} + e\sin^2\frac{\varphi}{2}\mathrm{J}_{ie}^{(i+1)} - \left(\sin^2\frac{\varphi}{2} - \frac{1}{4}e^2\right)\mathrm{J}_{ie}^{(i+2)}\right]$$

J denoting the Besselian function in Hansen's notation and $\sin\varphi = e$.

By substituting the preceding values the two last terms of R become

$$\frac{3}{4}\frac{m''}{m'}m'\frac{a^2}{a'^3}(1 - \gamma^2)^2\left\{\frac{a'^3(1 - \gamma''^2)^2}{\triangle^5}r''^2\Sigma.\ \mathrm{H}^{(i)}\cos(2h + 2g + il - 2v'' - 2h'')\right.$$
$$+ 2\frac{a'^3(1 - \gamma''^2)}{\triangle^5}r''r'\Sigma.\mathrm{H}^{(i)}\cos(2h + 2g + il - v'' - h'' - v' - h')$$
$$\left.+\frac{a'^3 r'^2}{\triangle^5}\Sigma.\mathrm{H}^{(i)}\cos(2h + 2g + il - 2v' - 2h')\right\}$$

If we suppose that

$$\triangle^{-5} = \frac{1}{2}\mathrm{B}^{(0)} + \mathrm{B}^{(1)}\cos(v'' + h'' - v' - h') + \mathrm{B}^{(2)}\cos 2(v'' + h'' - v' - h') + \ \ldots$$

and also put

$$\mathrm{C}^{(j)} = (1 - \gamma''^2)^2 r''^2 \mathrm{B}^{(j)} + 2(1 - \gamma''^2) r''r' \mathrm{B}^{(j-1)} + r'^2 \mathrm{B}^{(j-2)}$$

the foregoing expression takes the form

$$\frac{3}{8}m''a^2(1 - \gamma^2)^2\Sigma.\mathrm{C}^{(j)}\mathrm{H}^{(i)}\cos[2h + 2g + il - 2v'' - 2h'' + j(v'' + h'' - v' - h')]$$

where in the summation j as well as i must receive all integral values, negative and positive.

Let it be proposed to develop this expression in powers of e'' the eccentricity of Jupiter's orbit. As it is unnecessary to go beyond e''^2, we can put

$$\frac{r''}{a''} = 1 + \frac{1}{2}e''^2 - e''\cos l'' - \frac{1}{2}e''^2\cos 2l''$$
$$v'' = g'' + l'' + 2e''\sin l'' + \frac{5}{4}e''^2\sin 2l''$$

and preserve only those terms whose arguments contain $-2l''$. In this connection it will

be seen that it is unnecessary to consider any terms whose arguments contain any multiple of ν' beyond the single, since all of DELAUNAY's operations involving the argument l' have, at least, the factor $\frac{n'}{n}$, and thus the resulting terms would be of higher orders than we propose to consider. Hence it will suffice to consider only the values $j=0$, $j=-1$, and $j=+1$. Supposing that in $C^{(j)}$ we replace r'' by a'' our expression becomes

$$\begin{aligned}\frac{3}{8}m''a^2(1-\gamma^2)^2\Big\{\Sigma\,.\,&\left[(1-4e''^2)\,C^{(0)}+\left(\frac{1}{2}a''\frac{dC^{(0)}}{da''}+\frac{1}{4}a''^2\frac{d^2C^{(0)}}{da''^2}\right)e''^2\right]H^{(i)}\\&\qquad\times\cos(2h+2g+il-2h''-2g''-2l'')\\+\Sigma\,.\,&\left[-3C^{(-1)}-\frac{1}{2}a''\frac{dC^{(-1)}}{da''}\right]e''H^{(i)}\cos(2h+2g+il-3h''-3g''-2l''+\nu'+h')\\+\Sigma\,.\,&\left[C^{(1)}-\frac{1}{2}a''\frac{dC^{(1)}}{da''}\right]e''H^{(i)}\cos(2h+2g+il-h''-g''-2l''-\nu'-h')\Big\}\end{aligned}$$

In the next place this expression must be developed in powers of e', the eccentricity of the earth's orbit. It will suffice to put

$$\frac{r'}{a'}=1+\frac{1}{2}e'^2-e'\cos l'$$
$$\nu'=g'+l'+2e'\sin l'$$

and, for the reason just stated, preserve only the terms whose arguments are free from l'. Then, supposing that in $C^{(j)}$, r'', and r' are severally replaced by a'' and a', we have

$$\begin{aligned}\frac{3}{8}m''a^2(1-\gamma^2)^2\Big\{\Sigma\,.\,&\left[(1-4e''^2)C^{(0)}+\left(\frac{1}{2}a'\frac{dC^{(0)}}{da'}+\frac{1}{4}a'^2\frac{d^2C^{(0)}}{da'^2}\right)e'^2+\left(\frac{1}{2}a''\frac{dC^{(0)}}{da''}+\frac{1}{4}a''^2\frac{d^2C^{(0)}}{da''^2}\right)e''^2\right]\\&\qquad\times H^{(i)}\cos(2h+2g+il-2h''-2g''-2l'')\\+\Sigma\,.\,&\left[3C^{(-1)}+\frac{3}{2}a'\frac{dC^{(-1)}}{da'}+\frac{1}{2}a''\frac{dC^{(-1)}}{da''}+\frac{1}{4}a'a''\frac{d^2C^{(-1)}}{da'da''}\right]e'e''H^{(i)}\\&\qquad\times\cos(2h+2g+il-3h''-3g''-2l''+h'+g')\\+\Sigma\,.\,&\left[-C^{(1)}-\frac{1}{2}a'\frac{dC^{(1)}}{da'}+\frac{1}{2}a''\frac{dC^{(1)}}{da''}+\frac{1}{4}a'a''\frac{d^2C^{(1)}}{da'da''}\right]e'e''H^{(i)}\\&\qquad\times\cos(2h+2g+il-h''-g''-2l''-h'-g')\Big\}\end{aligned}$$

The effect of the inclination of Jupiter's orbit to the ecliptic on the value of $C^{(0)}$ can be in great part taken account of by equating the argument α the ratio of the mean distances. Thus, if we take

$$a''^2+a'^2=\mathrm{a}''^2+\mathrm{a}'^2$$
$$(1-\gamma''^2)\,a''a'=\mathrm{a}''\mathrm{a}'$$

we shall have

$$\triangle_0^2=\mathrm{a}''^2+2\mathrm{a}''\mathrm{a}'\cos\theta+\mathrm{a}'^2$$
$$\mathrm{a}''=a''\left(1+\gamma''^2\frac{\alpha^2}{1-\alpha^2}\right)$$
$$\mathrm{a}'=a'\left(1-\gamma''^2\frac{1}{1-\alpha^2}\right)$$

and, in determining the $b_s^{(i)}$, instead of the argument α, we ought to use $\alpha\left(1-\gamma''^2\frac{1+\alpha^2}{1-\alpha^2}\right)$.

Then we shall have

$$C^{(0)}=\frac{1}{a''^3}\left[b_{\frac{5}{2}}^{(0)}-2\alpha b_{\frac{5}{2}}^{(1)}+\alpha^2 b_{\frac{5}{2}}^{(2)}-\frac{2\gamma''^2}{1-\alpha^2}(b_{\frac{5}{2}}^{(0)}-\alpha^2 b_{\frac{5}{2}}^{(2)})\right]$$

$$=\frac{1}{a''^3}\left[b_{\frac{3}{2}}^{(0)}-\frac{2}{3}\alpha b_{\frac{3}{2}}^{(1)}-\frac{2\gamma''^2}{1-\alpha^2}(b_{\frac{5}{2}}^{(0)}-\alpha^2 b_{\frac{5}{2}}^{(2)})\right]$$

$$C^{(-1)}=\frac{1}{a''^3}\left[-b_{\frac{5}{2}}^{(1)}+2\alpha b_{\frac{5}{2}}^{(2)}-\alpha^2 b_{\frac{5}{2}}^{(3)}\right]$$

$$=\frac{1}{a''^3}\left[-4\alpha b_{\frac{3}{2}}^{(0)}+(1+\frac{8}{3}\alpha^2)\,b_{\frac{3}{2}}^{(1)}\right]$$

$$C^{(1)}=\frac{1}{a''^3}\left[-b_{\frac{5}{2}}^{(1)}+2\alpha b_{\frac{5}{2}}^{(0)}-\alpha^2 b_{\frac{5}{2}}^{(1)}\right]=-\frac{1}{3}\frac{1}{a''^3}b_{\frac{3}{2}}^{(1)}$$

The expression we have derived is simplified by taking the derivatives of the C with respect to α. Thus we get

$$\frac{3}{8}m''a^2(1-\gamma^2)^2\Big\{\Sigma\,.\left[C^{(0)}+\left(\frac{1}{2}\alpha\frac{dC^{(0)}}{d\alpha}+\frac{1}{4}\alpha^2\frac{d^2C^{(0)}}{d\alpha^2}\right)e'^2+\left(-\frac{5}{2}C_0+\frac{3}{2}\alpha\frac{dC^{(0)}}{d\alpha}+\frac{1}{4}\alpha^2\frac{d^2C^{(0)}}{d\alpha^2}\right)e''^2\right]$$
$$\times\,H^{(i)}\cos(2h+2g+il-2h''-2g''-2l'')$$
$$+\,\Sigma\,.\left[\frac{3}{2}C^{(-1)}-\frac{1}{4}\alpha^2\frac{d^2C^{(-1)}}{d\alpha^2}\right]e'e''\,H^{(i)}$$
$$\times\cos(2h+2g+il-3h''-3g''-2l''+h'+g')$$
$$-\,\Sigma\,.\left[\frac{5}{2}C^{(1)}+2\alpha\frac{dC^{(1)}}{d\alpha}+\frac{1}{4}\alpha^2\frac{d^2C^{(1)}}{d\alpha^2}\right]e'e''\,H^{(i)}$$
$$\times\cos(2h+2g+il-h''-g''-2l''-h'-g')\Big\}$$

Since α is quite small, the readiest method of obtaining the values of the factors of the coefficients in this expression which depend on it is by expansions in ascending powers of α. From the series for the $b_s^{(i)}$ given in the books we find

$$b_{\frac{3}{2}}^{(0)}-\frac{2}{3}\alpha b_{\frac{3}{2}}^{(1)}=2\left[1+\frac{1}{2}\cdot\frac{5}{2}\alpha^2+\frac{1\cdot3}{2\,.\,4}\cdot\frac{5\cdot7}{2\,.\,4}\alpha^4+\frac{1\cdot3\cdot5}{2\,.\,4\,.\,6}\cdot\frac{5\cdot7\cdot9}{2\,.\,4\,.\,6}\alpha^6\right.$$
$$\left.+\frac{1\cdot3\cdot5\cdot7}{2\,.\,4\,.\,6\,.\,8}\cdot\frac{5\cdot7\cdot9\cdot11}{2\,.\,4\,.\,6\,.\,8}\alpha^8+\ldots\right]$$

$$b_{\frac{5}{2}}^{(0)}-\alpha^2 b_{\frac{5}{2}}^{(2)}=2\left[1+\frac{5}{2}\cdot\frac{5}{2}\alpha^2+\frac{3\cdot9}{2\,.\,4}\cdot\frac{5\cdot7}{2\,.\,4}\alpha^4+\frac{3\cdot5\cdot13}{2\,.\,4\,.\,6}\cdot\frac{5\cdot7\cdot9}{2\,.\,4\,.\,6}\alpha^6\right.$$
$$\left.+\frac{3\cdot5\cdot7\cdot17}{2\,.\,4\,.\,6\,.\,8}\cdot\frac{5\cdot7\cdot9\cdot11}{2\,.\,4\,.\,6\,.\,8}\alpha^8+\ldots\right]$$

$$-4\alpha b_{\frac{3}{2}}^{(0)}+(1+\frac{8}{3}\alpha^2)\,b_{\frac{3}{2}}^{(1)}=-5\alpha\left[1+\frac{1}{2}\cdot\frac{7}{4}\alpha^2+\frac{1\cdot3}{2\,.\,4}\cdot\frac{7\cdot9}{4\,.\,6}\alpha^4+\frac{1\cdot3\cdot5}{2\,.\,4\,.\,6}\cdot\frac{7\cdot9\cdot11}{4\,.\,6\,.\,8}\alpha^6\right.$$
$$\left.+\frac{1\cdot3\cdot5\cdot7}{2\,.\,4\,.\,6\,.\,8}\cdot\frac{7\cdot9\cdot11\cdot13}{4\,.\,6\,.\,8\,.\,10}\alpha^8+\ldots\right]$$

$$-\frac{1}{3}b_{\frac{3}{2}}^{(1)}=-2\alpha\left[\frac{1}{2}+\frac{1\cdot3}{2\,.\,4}\cdot\frac{5}{2}\alpha^2+\frac{1\cdot3\cdot5}{2\,.\,4\,.\,6}\cdot\frac{5\cdot7}{2\,.\,4}\alpha^4+\frac{1\cdot3\cdot5\cdot7}{2\,.\,4\,.\,6\,.\,8}\cdot\frac{5\cdot7\cdot9}{2\,.\,4\,.\,6}\alpha^6+\ldots\right]$$

The inclination of Jupiter's orbit being $1^\circ\ 18'\ 42''$, we have $\log \gamma''^2 = 6.1173$. Also without correction $\log \alpha = 9.28376$, after correction $\log \alpha = 9.28370$. Thence we derive

$$C^{(0)} = 2.0963\,\frac{1}{a''^3},\quad \alpha\frac{dC^{(0)}}{d\alpha} = 0.2038\,\frac{1}{a''^3},\quad \alpha^2\frac{d^2C^{(0)}}{d\alpha^2} = 0.2446\,\frac{1}{a''^3},\quad C^{(-1)} = -0.9934\,\frac{1}{a''^3}$$

$$\alpha^2\frac{d^2C^{(-1)}}{d\alpha^2} = -0.2145\,\frac{1}{a''^3},\quad C^{(1)} = -0.2063\,\frac{1}{a''^3},\quad \alpha\frac{dC^{(1)}}{d\alpha} = -0.2360\,\frac{1}{a''^3},\quad \alpha^2\frac{d^2C^{(1)}}{d\alpha^2} = -0.0957\,\frac{1}{a''^3}$$

We will also put

$$e' = 0.01677, \qquad e'' = 0.04826, \qquad h'' + g'' - h' - g' = 91^\circ\ 33'$$

Employing Bessel's value of the mass of Jupiter, or $\frac{m''}{m'} = \frac{1}{1047.879}$, and expressing the coefficients in seconds of arc, our expression becomes

$$\frac{m'}{a'^3}a^2(1-\gamma^2)^2\Big\{\Sigma\,.\,1''.0928\,H^{(i)}\cos(2h + 2g + il - 2h'' - 2g'' - 2l'')$$

$$-\,0''.0010\,H^{(0)}\sin(2h + 2g - 2h'' - 2g'' - 2l'')\Big\}$$

The term of R, which was determined first, when reduced in a manner similar to this, has the expression

$$\frac{m'}{a'^3}\,0''.0517\,r^2\cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l')$$

where

$$r^2 = a^2\Big[\underset{[0]}{1 + \frac{3}{2}e^2} - \underset{[1]}{\left(2e - \frac{1}{4}e^3\right)\cos l} - \underset{[2]}{\frac{1}{2}e^2\cos 2l} - \underset{[3]}{\frac{1}{4}e^3\cos 3l}\Big]$$

We are now in possession of a suitable expression for R when elliptic values are attributed to the moon's co-ordinates. The effect of the solar perturbations must now be considered. If the transformations denoted by Delaunay as Operations 3, 4, 26, 40, and 41, are made in the terms of r^2, and only terms having the argument $2h + 2g - 2h' - 2g' - 2l'$ preserved, we find that r^2 contains the additional terms

$$a^2\Big\{\underset{[3\ldots\ldots3]}{-\frac{3}{16}e^2\frac{n'^2}{n^2}} + \underset{[4\ldots\ldots\ldots1]}{\frac{165}{8}e^2\frac{n'^2}{n^2}} - \underset{[26\ldots\ldots2]}{\frac{15}{16}e^2\frac{n'^2}{n^2}} + \underset{[40\ldots\ldots1]}{\frac{21}{16}e^2\frac{n'^2}{n^2}} + \underset{[41\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots0]}{\frac{45}{8}e^2\frac{n'}{n} + \frac{135}{32}e^2\frac{n'^2}{n^2}}\Big\}$$

$$\times\cos(2h + 2g - 2h' - 2g' - 2l')$$

$$= a^2\left[\frac{45}{8}e^2\frac{n'}{n} + \frac{801}{32}e^2\frac{n'^2}{n^2}\right]\cos(2h + 2g - 2h' - 2g' - 2l')$$

In the portion of R whose terms are factored by $H^{(i)}$, it is found necessary to attribute to i the values -1, 0, 1, 2, and 3. As no power of e above e^2 need be retained, the following is a sufficient expression for $H^{(i)}$:

$$H^{(i)} = \frac{2}{i}\left[\left(1 - \frac{1}{2}e^2\right)J_{\frac{ie}{2}}^{(i-2)} - \left(e - \frac{1}{4}e^3\right)J_{\frac{ie}{2}}^{(i-1)} + \frac{1}{4}e^2 J_{\frac{ie}{2}}^{(i+1)}\right]$$

with the understanding that $H^{(0)} = \frac{5}{2}e^2$, or these quantities may be taken from Professor CAYLEY's tables.*

Including the factor a^2, which is necessary in making the transformations, the five terms, written at length, are

$$(1) \qquad a^2 \Big\{ -\frac{7}{24} e^3 \cos\ 2h + 2g - l - 2h'' - 2g'' - 2l'')$$

$$(2) \qquad + \frac{5}{2} e^2 \cos (2h + 2g - 2h'' - 2g'' - 2l'')$$

$$(3) \qquad + \Big[-3e + \frac{13}{8} e^3 \Big] \cos (2h + 2g + l - 2h'' - 2g'' - 2l'')$$

$$(4) \qquad + \Big[1 - \frac{5}{2} e^2 \Big] \cos (2h + 2g + 2l - 2h'' - 2g'' - 2l'')$$

$$(5) \qquad + \Big[e - \frac{19}{8} e^3 \Big] \cos (2h + 2g + 3l - 2h'' - 2g'' - 2l'') \Big\}$$

The only operations which produce terms that we need retain are those numbered 2, 32, and 38, by DELAUNAY. These new terms, with the designation of their origin in the manner of DELAUNAY, are

$$a^2 \Big\{ \underset{[2 \ldots\ldots 1]}{\frac{7}{16} e^2 \frac{n'^2}{n^2}} + \underset{[2 \ldots\ldots 3]}{\frac{55}{8} e^2 \frac{n'^2}{n^2}} - \underset{[32 \ldots\ldots 4]}{\frac{5}{16} e^2 \frac{n'^2}{n^2}} - \underset{[38 \ldots\ldots 5]}{\frac{1}{16} e^2 \frac{n'^2}{n^2}} \Big\} \cos (2h + 2g - 2h'' - 2g'' - 2l'')$$

$$= \frac{111}{16} a^2 e^2 \frac{n'^2}{n^2} \cos (2h + 2g - 2h'' - 2g'' - 2l'')$$

When these terms, arising from solar perturbation, are joined to the elliptic value, the complete value of R, as far as it arises from the direct action of Jupiter (no terms but those involving the argument $2h + 2g - 2h'' - 2g'' - 2l''$ need now be retained), is

$$R = m' \frac{a^2}{a'^3} \Big\{ \Big[2''.732\, e^2 - 5''.46\, \gamma^2 e^2 + 0''.145\, e^2 \frac{n'}{n} + 8''.23\, e^2 \frac{n'^2}{n^2} \Big]$$

$$\times \cos (2h + 2g - 2h'' - 2g'' - 2l'')$$

$$- 0''.0025\, e^2 \sin (2h + 2g - 2h'' - 2g'' - 2l'') \Big\}$$

* Memoirs of the Royal Astronomical Society, Vol. XXIX.

II.—Terms of the Perturbative Function Arising from the Indirect Action of Jupiter.

We now consider the action of Jupiter in changing the solar perturbations of the moon. If R now denote the portion of the perturbative function produced by the action of the sun, and $\delta r'$, $\delta V'$, and $\delta U'$ the perturbations severally of the radius vector, longitude, and latitude of the sun by Jupiter, it is evident we ought to add to the expression of R, derived without regard to these perturbations, the expression

$$\delta R = \frac{dR}{dr'}\delta r' + \frac{dR}{dV'}\delta V' + \frac{dR}{dU'}\delta U'$$

But it is obvious the last term of this expression, when we restrict ourselves to the first power of Jupiter's mass, can give rise only to terms involving an odd multiple of h, the longitude of the moon's node. Consequently it may be neglected. As R only involves r' through the factor r'^{-3}, and at the same time is a function of $V - V'$, we may write

$$\delta R = -3R\frac{\delta r'}{r'} - \frac{dR}{dV}\delta V'$$

The parts of R and $\frac{dR}{dV}$ we need can be very readily obtained from the expansion of R given by Delaunay;* for it is found that the terms added to R by the solar perturbations, and which ought to be taken into account, arise from the five combinations in Delaunay's notation [2 ... 116], [2 ... 134], [3 ... 23], [26 ... 16], and [49 ... 166]. Now, it is found that no portion of the terms denoted by the latter number had been removed from the perturbative function when the operation designated by the first number was made in it. Hence we can copy immediately from Delaunay the terms we need; they are those numbered by him (125), (126), and (130):

$$R = m'\frac{a^2}{a'^3}\left\{\frac{15}{8}e^2 - \frac{15}{4}\gamma^2e^2 - \frac{75}{16}e^2e'^2 + \underset{[2.....116]}{\frac{165}{32}e^2\frac{n'^2}{n^2}} + \underset{[2.....134]}{\frac{21}{64}e^2\frac{n'^2}{n^2}} - \underset{[3.....23]}{\frac{3}{64}e^2\frac{n'^2}{n^2}} - \underset{[26.....16]}{\frac{15}{64}e^2\frac{n'^2}{n^2}} + \underset{[49...166]}{\frac{15}{8}\gamma^2e^2}\right\}$$

$$\times \cos(2h + 2g - 2h' - 2g' - 2l')$$

$$+ m'\frac{a^2}{a'^3}\left\{\frac{105}{16}e^2e'\right\}\cos(2h + 2g - 2h' - 2g' - 3l')$$

$$+ m'\frac{a^2}{a'^3}\left\{-\frac{15}{16}e^2e'\right\}\cos(2h + 2g - 2h' - 2g' - l')$$

$$= m'\frac{a^2}{a'^3}\left\{\frac{15}{8}e^2 - \frac{15}{8}\gamma^2e^2 - \frac{75}{16}e^2e'^2 + \frac{333}{64}e^2\frac{n'^2}{n^2}\right\}\cos(2h + 2g - 2h' - 2g' - 2l')$$

$$+ m'\frac{a^2}{a'^3}\left\{\frac{105}{16}e^2e'\right\}\cos(2h + 2g - 2h' - 2g' - 3l')$$

$$+ m'\frac{a^2}{a'^3}\left\{-\frac{15}{16}e^2e'\right\}\cos(2h + 2g - 2h' - 2g' - l')$$

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* Théorie du Mouvement de la Lune, Tom. J, pp. 119-256.

The proper expression for $\frac{dR}{dV}$ can be with ease obtained from the foregoing one for R by differentiating it partially with reference to D, that is, we multiply the coefficients by -2, and substitute sin for cos; but we must be careful to omit the two terms designated by the marks [3 ... 23] and [26 ... 16], for the reason that the terms numbered (16) and (23) do not contain D in their arguments. In this manner we get

$$\frac{dR}{dV} = m'\frac{a^2}{a'^3}\left\{-\frac{15}{4}e^2 + \frac{15}{4}\gamma^2 e^2 + \frac{75}{8}e^2 e'^2 - \frac{351}{32}e^2\frac{n'^2}{n^2}\right\}\sin(2h + 2g - 2h' - 2g' - 2l')$$

$$+ m'\frac{a^2}{a'^3}\left\{-\frac{105}{8}e^2 e'\right\}\sin(2h + 2g - 2h' - 2g' - 3l')$$

$$+ m'\frac{a^2}{a'^3}\left\{\frac{15}{8}e^2 e'\right\}\sin(2h + 2g - 2h' - 2g' - l')$$

In the next place we must have the values of the other factors $\delta r'$ and $\delta V'$. These we take from LEVERRIER.* After augmenting the coefficients by about 1–500th, in order to make them correspond to BESSEL's mass of Jupiter, the terms of LEVERRIER's expressions we need, become

$$\begin{aligned}
\delta V' = &- 2''.730 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l') \\
&+ 0''.014 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l') \\
&- 0''.020 \sin(2h'' + 2g'' + 2l'' - 3h' - 3g' - 3l') \\
&+ 0''.065 \cos(2h'' + 2g'' + 2l'' - 3h' - 3g' - 3l') \\
&- 0''.878 \sin(2h'' + 2g'' + 2l'' - h' - g' - l') \\
&- 1''.354 \cos(2h'' + 2g'' + 2l'' - h' - g' - l')
\end{aligned}$$

$$\begin{aligned}
\frac{\delta r'}{a'} = &- 1''.907 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l') \\
&- 0''.004 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l') \\
&- 0''.009 \cos(2h'' + 2g'' + 2l'' - 3h' - 3g' - 3l') \\
&- 0''.031 \sin(2h'' + 2g'' + 2l'' - 3h' - 3g' - 3l') \\
&- 0''.374 \cos(2h'' + 2g'' + 2l'' - h' - g' - l') \\
&+ 0''.567 \sin(2h'' + 2g'' + 2l'' - h' - g' - l')
\end{aligned}$$

By taking

$$h' + g' = 280^\circ\ 22'$$

and

$$\frac{a'}{r'} = 1 + 0.01677 \cos l'$$

we have, in a shape more suitable for our purposes,

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* Annales de l'Observatoire de Paris, Mémoires, Tom. IV, pp. 36, 37.

$$\begin{aligned}\delta V' = &- 2''.730 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l')\\ &+ 0''.014 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l')\\ &- 0''.068 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - 3l')\\ &- 0''.008 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 3l')\\ &- 1''.490 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - l')\\ &+ 0''.620 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - l')\end{aligned}$$

$$\begin{aligned}\frac{\delta r'}{r'} = &- 1''.912 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l')\\ &- 0''.006 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - 2l')\\ &- 0''.048 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - 3l')\\ &+ 0''.003 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - 3l')\\ &- 0''.641 \cos(2h'' + 2g'' + 2l'' - 2h' - 2g' - l')\\ &- 0''.266 \sin(2h'' + 2g'' + 2l'' - 2h' - 2g' - l')\end{aligned}$$

Multiplying the expressions for the factors together, and, for brevity, writing θ for the argument $2h + 2g - 2h'' - 2g'' - 2l''$, we get

$$\begin{aligned}-\frac{dR}{dV}\delta V' = m'\frac{a^2}{a'^3}\Big\{ &1''.365 \left[-\frac{15}{4}e^2 + \frac{15}{4}\gamma^2e^2 + \frac{75}{8}e^2e'^2 - \frac{351}{32}e^2\frac{n'^2}{n^2}\right]\cos\theta - 0''.007\left[-\frac{15}{4}e^2\right]\sin\theta\\ &+ 0''.034 \left[-\frac{105}{8}e^2e'\right]\cos\theta + 0''.004\left[-\frac{105}{8}e^2e'\right]\sin\theta + 0''.0745\left[\frac{15}{8}e^2e'\right]\cos\theta\\ &- 0''.310 \left[\frac{15}{8}e^2e'\right]\sin\theta \Big\}\end{aligned}$$

$$\begin{aligned}-3R\frac{\delta r'}{r'} = m'\frac{a^2}{a'^3}\Big\{ &2''.868 \left[\frac{15}{8}e^2 - \frac{15}{8}\gamma^2e^2 - \frac{75}{16}e^2e'^2 + \frac{333}{64}e^2\frac{n'^2}{n^2}\right]\cos\theta - 0''.009\left[\frac{15}{8}e^2\right]\sin\theta\\ &+ 0''.072 \left[\frac{105}{16}e^2e'\right]\cos\theta + 0''.005\left[\frac{105}{16}e^2e'\right]\sin\theta + 0''.961\left[-\frac{15}{16}e^2e'\right]\cos\theta\\ &- 0''.399 \left[-\frac{15}{16}e^2e'\right]\sin\theta\Big\}\end{aligned}$$

Attributing to e' its value 0.01677, the addition of the terms gives

$$\begin{aligned}\delta R = m'\frac{a^2}{a'^3}\Big\{ &\left[0''.267\, e^2 - 0''.26\,\gamma^2e^2 - 0''.05\, e^2\frac{n'^2}{n^2}\right]\cos(2h + 2g - 2h'' - 2g'' - 2l'')\\ &+ 0''.006\, e^2 \sin(2h + 2g - 2h'' - 2g'' - 2l'')\Big\}\end{aligned}$$

It will be seen in this result how the several terms have nearly canceled each other, and hence the indirect action augments the direct by a tenth part only.

III.—Integration of the Differential Equations by the Method of Delaunay.

Adding the portions of R which result severally from the direct and indirect actions of Jupiter we have as the complete expression to be employed in this research

$$R = m'\frac{a^2}{a'^3}\left\{\left[2''.999\, e^2 - 5''.72\, \gamma^2 e^2 + 0''.145\, e^2\frac{n'}{n} + 8''.18\, e^2\frac{n'^2}{n^2}\right]\cos(2h + 2g - 2h'' - 2g'' - 2l'')\right.$$
$$\left. + 0''.003\, e^2 \sin(2h + 2g - 2h'' - 2g'' - 2l'')\right\}$$

The term of this expression, which involves the sine of the argument, is so small that it may be neglected. Its only effect would be to change the argument of the inequalities by a few minutes of arc.

The signification of the symbols a, n, e, and γ in this expression are those of Delaunay before the transformation of Tom. II, p. 800 was made. From the data given by Delaunay we conclude that the numerical values are

$$\gamma = 0.04499 \qquad e = 0.05486 \qquad \frac{n'}{n} = 0.07440$$

Substituting these in the expression for R and its derivatives

$$R = 0''.00005072\, a^2n^2 \cos(2h + 2g - 2h'' - 2g'' - 2l'')$$
$$e\frac{dR}{de} = 0''.00010144\, a^2n^2 \cos(2h + 2g - 2h'' - 2g'' - 2l'')$$
$$a\frac{dR}{da} = 0''.00010393\, a^2n^2 \cos(2h + 2g - 2h'' - 2g'' - 2l'')$$
$$\gamma\frac{dR}{d\gamma} = -0''.00000039\, a^2n^2 \cos(2h + 2g - 2h'' - 2g'' - 2l'')$$

In all cases where the square of the disturbing force can be neglected, it appears to me that Delaunay's formulæ for integration are by far the least laborious that have been proposed; especially is this the case when we are content with numerical values for the coefficients. Then certain auxiliary quantities in Delaunay's formulæ, which are the same whatever the inequality considered, may be at once reduced to their numerical values. Hence it seems worth while to develop this method of proceeding in a general manner, so that it may be applicable to any case that may arise.

Employing n to denote the mean angular motion of the moon, equivalent in Delaunay's notation to $h_0 + g_0 + l_0$, the differential equations, which the augmentations of the six quantities a, e, γ, l, g, and h satisfy, are

$$\frac{d.\delta a}{dt} = \frac{da}{dL}\frac{dR}{dl} + \frac{da}{dG}\frac{dR}{dg} + \frac{da}{dH}\frac{dR}{dh}$$
$$\frac{d.\delta e}{dt} = \frac{de}{dL}\frac{dR}{dl} + \frac{de}{dG}\frac{dR}{dg} + \frac{de}{dH}\frac{dR}{dh}$$
$$\frac{d.\delta \gamma}{dt} = \frac{d\gamma}{dL}\frac{dR}{dl} + \frac{d\gamma}{dG}\frac{dR}{dg} + \frac{d\gamma}{dH}\frac{dR}{dh}$$

$$\frac{d.\delta(h+g+l)}{dt} = \frac{d\mathrm{n}}{dn}\delta n + \frac{d\mathrm{n}}{de}\delta e + \frac{d\mathrm{n}}{d\gamma}\delta\gamma - \left[\frac{da}{d\mathrm{L}} + \frac{da}{d\mathrm{G}} + \frac{da}{d\mathrm{H}}\right]\frac{d\mathrm{R}}{da} - \left[\frac{de}{d\mathrm{L}} + \frac{de}{d\mathrm{G}} + \frac{de}{d\mathrm{H}}\right]\frac{d\mathrm{R}}{de}$$
$$- \left[\frac{d\gamma}{d\mathrm{L}} + \frac{d\gamma}{d\mathrm{G}} + \frac{d\gamma}{d\mathrm{H}}\right]\frac{d\mathrm{R}}{d\gamma}$$

$$\frac{d.\delta l}{dt} = \frac{dl_0}{dn}\delta n + \frac{dl_0}{de}\delta e + \frac{dl_0}{d\gamma}\delta\gamma - \frac{da}{d\mathrm{L}}\frac{d\mathrm{R}}{da} - \frac{de}{d\mathrm{L}}\frac{d\mathrm{R}}{de} - \frac{d\gamma}{d\mathrm{L}}\frac{d\mathrm{R}}{d\gamma}$$

$$\frac{d.\delta h}{dt} = \frac{dh_0}{dn}\delta n + \frac{dh_0}{de}\delta e + \frac{dh_0}{d\gamma}\delta\gamma - \frac{da}{d\mathrm{H}}\frac{d\mathrm{R}}{da} - \frac{de}{d\mathrm{H}}\frac{d\mathrm{R}}{de} - \frac{d\gamma}{d\mathrm{H}}\frac{d\mathrm{R}}{d\gamma}$$

The analytical expressions for the quantities $\frac{da}{d\mathrm{L}}$, $\frac{da}{d\mathrm{G}}$, &c., are given by DELAUNAY,* and on substituting for γ, e, $\frac{n'}{n}$, &c., their numerical values which have been previously noted, we get

$$an\frac{da}{d\mathrm{L}} = 2.002730 \qquad an\frac{da}{d\mathrm{G}} = -0.003311 \qquad an\frac{da}{d\mathrm{H}} = -0.000084$$

$$a^2ne\frac{de}{d\mathrm{L}} = 1.0475 \qquad a^2ne\frac{de}{d\mathrm{G}} = -1.049176 \qquad a^2ne\frac{de}{d\mathrm{H}} = 0.000176$$

$$a^2n\gamma\frac{d\gamma}{d\mathrm{L}} = 0.000063 \qquad a^2n\gamma\frac{d\gamma}{d\mathrm{G}} = 0.24972 \qquad a^2n\gamma\frac{d\gamma}{d\mathrm{H}} = -0.25073$$

In the next place, by partial differentiation of the expressions for n, l_0, and h_0,† we obtain

$$\frac{d\mathrm{n}}{dn} = 1.00474\ddagger \qquad \frac{1}{n}\frac{d\mathrm{n}}{de} = -0.002076 \qquad \frac{1}{n}\frac{d\mathrm{n}}{d\gamma} = 0.002039$$

$$\frac{dl_0}{dn} = 1.01946 \qquad \frac{1}{n}\frac{dl_0}{de} = -0.001055 \qquad \frac{1}{n}\frac{dl_0}{d\gamma} = 0.006520$$

$$\frac{dh_0}{dn} = 0.003751 \qquad \frac{1}{n}\frac{dh_0}{de} = -0.001317 \qquad \frac{1}{n}\frac{dh_0}{d\gamma} = 0.000667$$

To all these quantities have been applied inductive corrections when the slowness of the convergence of the series appeared to require them.

We can write

$$\frac{d\mathrm{n}}{d\mathrm{L}} = -\frac{3}{2}\frac{n}{a}\frac{d\mathrm{n}}{dn}\frac{da}{d\mathrm{L}} + \frac{d\mathrm{n}}{de}\frac{de}{d\mathrm{L}} + \frac{d\mathrm{n}}{d\gamma}\frac{d\gamma}{d\mathrm{L}}$$

$$\frac{d\mathrm{n}}{d\mathrm{G}} = -\frac{3}{2}\frac{n}{a}\frac{d\mathrm{n}}{dn}\frac{da}{d\mathrm{G}} + \frac{d\mathrm{n}}{de}\frac{de}{d\mathrm{G}} + \frac{d\mathrm{n}}{d\gamma}\frac{d\gamma}{d\mathrm{G}}$$

$$\frac{d\mathrm{n}}{d\mathrm{H}} = -\frac{3}{2}\frac{n}{a}\frac{d\mathrm{n}}{dn}\frac{da}{d\mathrm{H}} + \frac{d\mathrm{n}}{de}\frac{de}{d\mathrm{H}} + \frac{d\mathrm{n}}{d\gamma}\frac{d\gamma}{d\mathrm{H}}$$

$$\frac{dl_0}{d\mathrm{L}} = -\frac{3}{2}\frac{n}{a}\frac{dl_0}{dn}\frac{da}{d\mathrm{L}} + \frac{dl_0}{de}\frac{de}{d\mathrm{L}} + \frac{dl_0}{d\gamma}\frac{d\gamma}{d\mathrm{L}}$$

$$\frac{dl_0}{d\mathrm{G}} = -\frac{3}{2}\frac{n}{a}\frac{dl_0}{dn}\frac{da}{d\mathrm{G}} + \frac{dl_0}{de}\frac{de}{d\mathrm{G}} + \frac{dl_0}{d\gamma}\frac{d\gamma}{d\mathrm{G}}$$

$$\frac{dl_0}{d\mathrm{H}} = -\frac{3}{2}\frac{n}{a}\frac{dl_0}{dn}\frac{da}{d\mathrm{H}} + \frac{dl_0}{de}\frac{de}{d\mathrm{H}} + \frac{dl_0}{d\gamma}\frac{d\gamma}{d\mathrm{H}}$$

* Tom. I, pp. 834, 835, 857, 858. † Tom. II, pp. 237, 238, 799.

‡ This number and those of the following which depend upon it have been rectified. I am indebted to M. R. Radau for indicating the necessity of this (*Recherches concernant les Inégalités du Mouvement de la Lune*).

$$\frac{dh_0}{dL} = -\frac{3}{2}\frac{n}{a}\frac{dh_0}{dn}\frac{da}{dL} + \frac{dh_0}{de}\frac{de}{dL} + \frac{dh_0}{d\gamma}\frac{d\gamma}{dL}$$

$$\frac{dh_0}{dG} = -\frac{3}{2}\frac{n}{a}\frac{dh_0}{dn}\frac{da}{dG} + \frac{dh_0}{de}\frac{de}{dG} + \frac{dh_0}{d\gamma}\frac{d\gamma}{dG}$$

$$\frac{dh_0}{dH} = -\frac{3}{2}\frac{n}{a}\frac{dh_0}{dn}\frac{da}{dH} + \frac{dh_0}{de}\frac{de}{dH} + \frac{dh_0}{d\gamma}\frac{d\gamma}{dH}$$

From these formulæ, in like manner, we obtain

$$a^2\frac{d\mathrm{n}}{dL} = -3.0580 \qquad a^2\frac{d\mathrm{n}}{dG} = 0.05601 \qquad a^2\frac{d\mathrm{n}}{dH} = -0.01124$$

$$a^2\frac{dl_0}{dL} = -3.0826 \qquad a^2\frac{dl_0}{dG} = 0.06142 \qquad a^2\frac{dl_0}{dH} = -0.03621$$

$$a^2\frac{dh_0}{dL} = -0.03641 \qquad a^2\frac{dh_0}{dG} = 0.02890 \qquad a^2\frac{dh_0}{dH} = -0.00372$$

Let us suppose that

$$R = A\cos(il + i'g + i''h + \nu t + q) = A\cos\theta$$

where ν denotes the portion of the motion of the argument which is independent of the mean motion of moon and of the motions of its perigee and node; q denotes a constant. The integrating factor we denote by μ; so that

$$\mu = \left[\frac{l_0}{\mathrm{n}}i + \frac{g_0}{\mathrm{n}}i' + \frac{h_0}{\mathrm{n}}i'' + \frac{\nu}{\mathrm{n}}\right]^{-1} = \left[0.991547996i + 0.012473741i' - 0.004021737i'' + \frac{\nu}{\mathrm{n}}\right]^{-1}$$

The value of n, the unit of time being the Julian year, is 17325594″.

We then have

$$\frac{\delta a}{a} = \left[i\frac{da}{dL} + i'\frac{da}{dG} + i''\frac{da}{dH}\right]\frac{\mu A}{a\mathrm{n}}\cos\theta$$

$$\delta e = \left[i\frac{de}{dL} + i'\frac{de}{dG} + i''\frac{de}{dH}\right]\frac{\mu A}{\mathrm{n}}\cos\theta$$

$$\delta\gamma = \left[i\frac{d\gamma}{dL} + i'\frac{d\gamma}{dG} + i''\frac{d\gamma}{dH}\right]\frac{\mu A}{\mathrm{n}}\cos\theta$$

$$\delta(h+g+l) = \mu\left\{\left[i\frac{d\mathrm{n}}{dL} + i'\frac{d\mathrm{n}}{dG} + i''\frac{d\mathrm{n}}{dH}\right]\frac{\mu A}{\mathrm{n}^2} - \frac{1}{\mathrm{n}}\left[\frac{da}{dL} + \frac{da}{dG} + \frac{da}{dH}\right]\frac{dA}{da}\right.$$

$$\left. - \frac{1}{\mathrm{n}}\left[\frac{de}{dL} + \frac{de}{dG} + \frac{de}{dH}\right]\frac{dA}{de} - \frac{1}{\mathrm{n}}\left[\frac{d\gamma}{dL} + \frac{d\gamma}{dG} + \frac{d\gamma}{dH}\right]\frac{dA}{d\gamma}\right\}\sin\theta$$

$$\delta l = \mu\left\{\left[i\frac{dl_0}{dL} + i'\frac{dl_0}{dG} + i''\frac{dl_0}{dH}\right]\frac{\mu A}{\mathrm{n}^2} - \frac{1}{\mathrm{n}}\frac{da}{dL}\frac{dA}{da} - \frac{1}{\mathrm{n}}\frac{de}{dL}\frac{dA}{de} - \frac{1}{\mathrm{n}}\frac{d\gamma}{dL}\frac{dA}{d\gamma}\right\}\sin\theta$$

$$\delta h = \mu\left\{\left[i\frac{dh_0}{dL} + i'\frac{dh_0}{dG} + i''\frac{dh_0}{dH}\right]\frac{\mu A}{\mathrm{n}^2} - \frac{1}{\mathrm{n}}\frac{da}{dH}\frac{dA}{da} - \frac{1}{\mathrm{n}}\frac{de}{dH}\frac{dA}{de} - \frac{1}{\mathrm{n}}\frac{d\gamma}{dH}\frac{dA}{d\gamma}\right\}\sin\theta$$

When the numerical values of the quantities which have been just determined are substituted in these equations, and the quantities a, e, and γ, which appear in the left members are made to have the signification which DELAUNAY attributes to them after the transformation of Tom. II, p. 800, we have

$$\frac{\delta a}{a} = \left[2.0135\, i - 0.003329\, i' - 0.000084 i''\right] \frac{\mu A}{a^2 n^2} \cos\theta$$

$$\delta e = \left[19.207\, i - 19.238\, i' + 0.0032\, i''\right] \frac{\mu A}{a^2 n^2} \cos\theta$$

$$\delta\gamma = \left[0.0014\, i + 5.5674\, i' - 5.5899\, i''\right] \frac{\mu A}{a^2 n^2} \cos\theta$$

$$\delta(h + g + l) = \frac{\mu}{a^2 n^2} \Big\{ \left[-3.0906\, i + 0.05661\, i' - 0.01136\, i''\right] \mu A - 2.0100\, a \frac{dA}{da}$$

$$+ 0.3447\, e \frac{dA}{de} + 0.4719\, \gamma \frac{dA}{d\gamma} \Big\} \sin\theta$$

$$\delta l = \frac{\mu}{a^2 n^2} \Big\{ \left[-3.1156\, i + 0.06208\, i' - 0.03660\, i''\right] \mu A - 2.0134\, a \frac{dA}{da}$$

$$- 349.84\, e \frac{dA}{de} - 0.0313\, \gamma \frac{dA}{d\gamma} \Big\} \sin\theta$$

$$\delta h = \frac{\mu}{a^2 n^2} \Big\{ \left[-0.03680\, i + 0.02921\, i' - 0.00376\, i'\right] \mu A + 0.00008\, a \frac{dA}{da}$$

$$- 0.05877\, e \frac{dA}{de} + 124.54\, \gamma \frac{dA}{d\gamma} \Big\} \sin\theta$$

In the special inequality we are dealing with $i = 0$, $i' = 2$, $i'' = 2$, $\mu = 233.0$. On substituting these values together with the proper values of A and its derivatives we get

$$\delta e = -0''.4546 \cos(2h + 2g - 2h'' - 2g'' - 2l'')$$

$$\delta(h + g + l) = +0''.2091 \sin(2h + 2g - 2h'' - 2g'' - 2l'')$$

$$e\delta l = -0''.4490 \sin(2h + 2g - 2h'' - 2g'' - 2l'')$$

The variations of the other elements are small enough to be neglected.

If these variations of the elements are made in the mean longitude, the principal term of the equation of the center and in the evection, we get as the perturbations of the true longitude

$$\delta V = -0''.903 \sin(2h + 2g + l - 2h'' - 2g'' - 2l'')$$

$$+ 0''.209 \sin(2h + 2g - 2h'' - 2g'' - 2l'')$$

$$- 0''.188 \sin(l - 2h' - 2g' - 2l' + 2h'' + 2g'' + 2l'')$$

These are all the terms which seem sufficiently large to be worthy of notice.

It will be perceived that the coefficients of the first and second differ from those given by Mr. NEISON, especially the latter, which is only about one-tenth of Mr. NEISON's value. On the cause of this disagreement it is impossible at present to pronounce, as Mr. NEISON has given no indication of the method he employed. Although I do not wish to be too positive in asserting the correctness of the foregoing investigation, as it is possible some oversight may have been committed, yet I may be allowed to say that great pains have been taken to avoid such. It is to be hoped that Mr. NEISON will shortly afford us the means of deciding this interesting matter.

IV.—TRANSFORMATION FORMULÆ OF DELAUNAY EMPLOYED IN THE PRECEDING INVESTIGATION.

In order to save reference to DELAUNAY's volumes, I will give the formulæ of transformation of DELAUNAY's operations so far as they are needed for the determination of the effect of solar perturbation in adding new terms to the coefficients of the inequalities here discussed.

Operation 2.

We replace

$$a \text{ by } a\{1 - e\frac{n'^2}{n^2}\cos l\}$$

$$e\cos l \text{ by } e\cos l + \frac{27}{16}e^2\frac{n'^2}{n^2}\cos 2l$$

$$e\sin l \text{ by } e\sin l + \frac{27}{16}e^2\frac{n'^2}{n^2}\sin 2l$$

$$h + g + l \text{ by } h + g + l + \frac{13}{4}e\frac{n'^2}{n^2}\sin l$$

$$e^2 \text{ by } e^2 - e\cos l$$

$$e^3\cos 3l \text{ by } e^3\cos 3l - \frac{3}{2}e^2\frac{n'^2}{n^2}\cos 2l$$

$$e^3\sin 3l \text{ by } e^3\sin 3l - \frac{3}{2}e^2\frac{n'^2}{n^2}\sin 2l$$

Operation 3.

We replace

$$e^3\cos 3(2h + 2g + 3l - 2h' - 2g' - 2l') \text{ by } e^3\cos 3(2h + 2g + 3l - 2h' - 2g' - 2l') + \frac{3}{4}e^2\frac{n'^2}{n^2}\cos 2(2h + 2g + 3l - 2h' - 2g' - 2l')$$

$$e^3\sin 3(2h + 2g + 3l - 2h' - 2g' - 2l') \text{ by } e^3\sin 3(2h + 2g + 3l - 2h' - 2g' - 2l') + \frac{3}{4}e^2\frac{n'^2}{n^2}\sin 2(2h + 2g + 3l - 2h' - 2g' - 2l')$$

Operation 4.

We replace

$$a \text{ by } a\{1 - \frac{9}{2} e \frac{n'^2}{n^2} \cos(2h + 2g + l - 2h' - 2g' - 2l')\}$$

$$e^2 \text{ by } e^2 + \frac{9}{2} e \frac{n'^2}{n^2} \cos(2h + 2g + l - 2h' - 2g' - 2l')$$

$$h + g + l \text{ by } h + g + l + \frac{117}{8} e \frac{n'^2}{n^2} \sin(2h + 2g + l - 2h' - 2g' - 2l')$$

$$e \cos(2h + 2g + l - 2h' - 2g' - 2l') \text{ by } e \cos(2h + 2g + l - 2h' - 2g' - 2l')$$
$$+ \frac{291}{32} e^2 \frac{n'^2}{n^2} \cos 2(2h + 2g + l - 2h' - 2g' - 2l')$$

$$e \sin(2h + 2g + l - 2h' - 2g' - 2l') \text{ by } e \sin(2h + 2g + l - 2h' - 2g' - 2l')$$
$$+ \frac{291}{32} e^2 \frac{n'^2}{n^2} \sin 2(2h + 2g + l - 2h' - 2g' - 2l')$$

Operation 26.

We replace

$$a \text{ by } a\{1 + \frac{3}{2} \frac{n'^2}{n^2} \cos(2h + 2g + 2l - 2h' - 2g' - 2l')\}$$

$$e^2 \text{ by } e^2 - \frac{3}{4} e^2 \frac{n'^2}{n^2} \cos(2h + 2g + 2l - 2h' - 2g' - 2l')$$

$$l \text{ by } l - \frac{3}{4} \frac{n'^2}{n^2} \sin(2h + 2g + 2l - 2h' - 2g' - 2l')$$

Operation 32.

We replace

$$a \text{ by } a\{1 - \frac{1}{4} e^2 \frac{n'^2}{n^2} \cos 2l\}$$

$$e^2 \text{ by } e^2 - \frac{1}{4} e^2 \frac{n'^2}{n^2} \cos 2l$$

$$h + g + l \text{ by } h + g + l + \frac{3}{8} e^2 \frac{n'^2}{n^2} \sin 2l$$

Operation 38.

We replace

$$e \text{ by } e - \frac{1}{16} e^2 \frac{n'^2}{n^2} \cos 3l$$

$$l \text{ by } l + \frac{1}{16} e \frac{n'^2}{n^2} \sin 3l$$

Operation 40.

We replace

$$e \text{ by } e - \frac{21}{32} e^2 \frac{n'^2}{n^2} \cos(2h + 2g - l - 2h' - 2g' - 2l')$$

$$l \text{ by } l - \frac{21}{32} e \frac{n'^2}{n^2} \sin(2h + 2g - l - 2h' - 2g' - 2l')$$

Operation 41.

We replace

$$e^2 \text{ by } e^2 + \left[\frac{15}{4} e^2 \frac{n'}{n} + \frac{45}{16} e^2 \frac{n'^2}{n^2}\right] \cos(2h + 2g - 2h' - 2g' - 2l')$$

Operation 49.

We replace

$$\gamma^2 \text{ by } \gamma^2 + \frac{5}{4} \gamma^2 e^2 \cos 2g$$

$$h \text{ by } h + \frac{5}{8} e^2 \sin 2g$$